ON THE ELASTIC SCATTERING OF THE FAST MESONS.

By S. GUPTA, *Department of Applied Mathematics, University of Calcutta.*

(Communicated by Prof. N. R. Sen, D.Sc., Ph.D., F.N.I.)

(*Received July 2, 1942 ; Read October 5, 1942.*)

ABSTRACT.

The differential cross-sections of the elastic scattering of mesons with spin one and zero are calculated from Duffin-Kemmer's β-formalism by a method which is analogous to the elegant method of Sauter as applied to Dirac's electron.

INTRODUCTION.

The problem of the elastic scattering of meson with spin one was discussed by various authors (Laporte, 1938; Massey and Corben, 1939; Majumdar and Gupta, 1941) assuming the field theory of meson obeying Proca's equation. Recently Duffin (1938) and Kemmer (1939) derived the equations of particles with spin one and zero in a form which is analogous to Dirac's equation. This formulation of Duffin-Kemmer, as shown by various authors (Booth and Wilson, 1940; Wilson, 1940; Christy and Kusaka, 1941) on performing the calculations by a method which is somewhat similar to that as applied to Dirac's electron, can be used with more advantage in the problems of the interaction between meson and the electromagnetic field than the usual field theory. The relativistic theory of the elastic scattering of charged particle with spin half was first worked out by Mott (1929) from the Dirac's theory of electron and the calculation was later on much simplified by an elegant method by Sauter (1933). In the latter method a Born's approximation was used and the scalar force field through which the colliding particle interacts with the nucleus was left as completely general. As this general field was not specialised to Coulomb field, until the final stage, it was possible to avoid all convergence difficulties. The purpose of this note is to calculate the differential cross-section of the elastic scattering of meson from the Duffin-Kemmer's formalism by a method which is analogous to the method of Sauter. By this procedure we can avoid the use of the method of second quantisation which is usually utilised in a similar problem of meson when using the field theory of Proca. Laporte, of course, applied to Proca's equation a treatment which is somewhat similar to Sauter's one; but his calculations were not so simple. The advantage of using the Duffin-Kemmer's formalism is that the cross-sections for particles with spin one and zero are obtained from a single scheme.

CALCULATIONS.

The meson is defined in the presence of the electromagnetic field by Kemmer by the equation

$$\left(\frac{\partial}{\partial x_\mu}-\frac{ie}{\hbar c}\phi_\mu\right)\beta_\mu\psi+\kappa\psi=0, \qquad (1)$$

where $\kappa=\frac{mc}{\hbar}$, $x_4=ict$, together with the following commutation rules for the Duffin's β-matrices

$$\beta_\mu\beta_\nu\beta_\rho+\beta_\rho\beta_\nu\beta_\mu=\beta_\mu\delta_{\nu\rho}+\beta_\rho\delta_{\nu\mu}\ldots \qquad (2)$$

The wave function ψ satisfies the initial condition

$$\left(\frac{\partial}{\partial x_k}-\frac{ie}{\hbar c}\phi_k\right)\beta_k\beta_4^2\,\psi+\kappa\left(1-\beta_4^2\right)\psi=0,\quad (k=1,2,3). \qquad (3)$$

The second order wave equation, which is satisfied by each component of ψ, is given by

$$\left(\frac{\partial}{\partial x_\mu}-\frac{ie}{\hbar c}\phi_\mu\right)\left(\frac{\partial}{\partial x_\mu}-\frac{ie}{\hbar c}\phi_\mu\right)\psi=\kappa^2\psi+\frac{ie}{2\hbar c}F_{\mu\nu}(\beta_\mu\beta_\nu-\beta_\nu\beta_\mu)\,\psi$$
$$+\frac{ie}{2mc^2}\left(\frac{\partial}{\partial x_\mu}-\frac{ie}{\hbar c}\phi_\mu\right)F_{\nu\rho}(\beta_\rho\beta_\mu\beta_\nu-\beta_\nu\delta_{\mu\rho})\,\psi. \qquad (4)$$

To be definite we shall consider in our subsequent discussion the case of positive meson, but our final result will also be valid for negative meson. In the absence of the magnetic field and for any arbitrary scalar potential function V, the equation (4) assumes the form

$$\left\{\nabla^2+\left(\frac{V-E}{\hbar c}\right)^2-\frac{m^2c^2}{\hbar^2}\right\}\psi=-\frac{1}{\hbar c}\frac{\partial V}{\partial x_k}(\beta_k\beta_4-\beta_4\beta_k)\,\psi$$
$$+\frac{1}{mc^2}\left(\frac{V-E}{\hbar c}\right)\frac{\partial V}{\partial x_k}\beta_k\beta_4^2\,\psi-\frac{1}{mc^2}\frac{\partial}{\partial x_l}\left(\frac{\partial V}{\partial x_k}\beta_4\beta_l\beta_k\,\psi\right),\quad (k,l=1,2,3),$$

or

$$\left(\nabla^2+\frac{p^2}{\hbar^2}\right)\psi=\frac{1}{\hbar c}\left\{\frac{2EV}{\hbar c}-\frac{\partial V}{\partial x_k}(\beta_k\beta_4-\beta_4\beta_k)-\frac{E}{mc^2}\frac{\partial V}{\partial x_k}\beta_k\beta_4^2\right\}\psi$$
$$-\frac{1}{mc^2}\frac{\partial}{\partial x_l}\left(\frac{\partial V}{\partial x_k}\beta_4\beta_l\beta_k\,\psi\right)+\frac{1}{\hbar c}\left\{-\frac{V^2}{\hbar c}+\frac{1}{mc^2}V\frac{\partial V}{\partial x_k}\beta_k\beta_4^2\right\}\psi.\ldots \qquad (5)$$

In this equation we regard V as a perturbing potential and assume ψ to be of the form

$$\psi=\psi_0+\psi_1+\psi_2+\ldots, \qquad (6)$$

where ψ_0 represents the wave function of the free meson and has the form

$$\psi_0=u^+(\mathbf{p})e^{\frac{i}{\hbar}(\mathbf{pr})}, \qquad (7)$$

and further from (1) and (3) we have

$$\{-E\beta_4+ic(\mathbf{p}\beta)+mc^2\}\,u^+(\mathbf{p})=0, \quad \ldots \quad \ldots \quad (8)$$

with the equation of condition

$$u^+(\mathbf{p})=\left\{1-\frac{i}{mc}(\mathbf{p}\beta)\right\}\beta_4^2u^+(\mathbf{p}). \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

From (5) and (6) one then obtains the recurrence formula for ψ_n as

$$\left(\nabla^2+\frac{p^2}{\hbar^2}\right)\psi_n=\frac{1}{\hbar c}\left\{\frac{2EV}{\hbar c}-\frac{\partial V}{\partial x_k}(\beta_k\beta_4-\beta_4\beta_k)-\frac{E}{mc^2}\frac{\partial V}{\partial x_k}\beta_k\beta_4^2\right\}\psi_{n-1}$$

$$-\frac{1}{mc^2}\frac{\partial}{\partial x_l}\left(\frac{\partial V}{\partial x_k}\beta_4\beta_l\beta_k\psi_{n-1}\right)+\frac{1}{\hbar c}\left(-\frac{V^2}{\hbar c}+\frac{1}{mc^2}V\frac{\partial V}{\partial x_k}\beta_k\beta_4^2\right)\psi_{n-2}. \quad \ldots \quad (10)$$

In this note, as we are interested only on the first order approximation, the last term on the right hand side of (5) may be neglected at large distances R from the scattering centre, and the equation for ψ_1 then becomes

$$\left(\nabla^2+\frac{p^2}{\hbar^2}\right)\psi_1=\frac{1}{\hbar c}\left\{\frac{2EV}{\hbar c}-\frac{\partial V}{\partial x_k}(\beta_k\beta_4-\beta_4\beta_k)-\frac{E}{mc^2}\frac{\partial V}{\partial x_k}\beta_k\beta_4^2\right\}\psi_0$$

$$-\frac{1}{mc^2}\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x_k}\beta_4\beta_l\beta_k\psi_0\right). \quad \ldots \quad (11)$$

The solution of this equation is well known and has the form

$$\psi_1=\frac{1}{4\pi\hbar c}\int\frac{e^{\frac{ip}{\hbar}|\mathbf{R}-\mathbf{r}|}}{|\mathbf{R}-\mathbf{r}|}\left[\left\{-\frac{2EV}{\hbar c}+\frac{\partial V}{\partial x_k}(\beta_k\beta_4-\beta_4\beta_k)+\frac{E}{mc^2}\frac{\partial V}{\partial x_k}\beta_k\beta_4^2\right\}\psi_0\right.$$

$$\left.+\frac{\hbar}{mc}\frac{\partial}{\partial x_l}\left(\frac{\partial V}{\partial x_k}\beta_4\beta_l\beta_k\psi_0\right)\right]d\tau, \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

where $\mathbf{R}$ ($=\xi_1,\ \xi_2,\ \xi_3$) is the space-vector of the reference point at which ψ_1 is to be calculated, and $\mathbf{r}$ ($=x_1,\ x_2,\ x_3$) is the integration point. The sign of the power of e is so chosen that (12) represents an outgoing spherical wave at a great distance from the scattering centre.

We integrate (12) by parts and neglect the surface integrals, and since

$$\frac{\partial}{\partial x_k}f(\mathbf{R}-\mathbf{r})=-\frac{\partial}{\partial \xi_k}f(\mathbf{R}-\mathbf{r}),$$

and ψ_0 is a function of $\mathbf{r}$ only, we have

$$\psi_1=\frac{1}{4\pi\hbar c}\int\left[-\frac{2EV}{\hbar c}+\frac{\partial V}{\partial x_k}\left\{(\beta_k\beta_4-\beta_4\beta_k)+\frac{E}{mc^2}\beta_k\beta_4^2\right.\right.$$

$$\left.\left.+\frac{\hbar}{mc}\beta_4\beta_l\beta_k\frac{\partial}{\partial \xi_l}\right\}\right]\psi_0\frac{e^{\frac{ip}{\hbar}|\mathbf{R}-\mathbf{r}|}}{|\mathbf{R}-\mathbf{r}|}\,d\tau \quad \ldots \quad (13)$$

To determine the scattering probability we are mainly concerned with the behaviour of this function at great distances from the scattering centre. We then have

$$|\mathbf{R}-\mathbf{r}| = R\left\{1+\frac{r^2}{R^2}-\frac{2}{R}(\mathbf{nr})\right\}^{\frac{1}{2}} = R-(\mathbf{nr}), \text{ approximately,}$$

where $\mathbf{n}$ denotes the unit vector in the direction of $\mathbf{R}$, that is in the direction of observation; further we introduce the unit vector $\mathbf{e} = \dfrac{\mathbf{p}}{|p|}$ in the primary direction of the meson. Hence for large R we have approximately by (7)

$$\int f(r)\,\psi_0\cdot\frac{e^{\frac{ip}{\hbar}|\mathbf{R}-\mathbf{r}|}}{|\mathbf{R}-\mathbf{r}|}\,d\tau = u^+(\mathbf{p})\int f(r)\cdot\frac{e^{\frac{ip}{\hbar}\{|\mathbf{R}-\mathbf{r}|+(\mathbf{er})\}}}{|\mathbf{R}-\mathbf{r}|}\,d\tau$$

$$\rightarrow u^+(\mathbf{p})\frac{e^{\frac{ip}{\hbar}R}}{R}\cdot\int f(r)\,e^{\frac{ip}{\hbar}(\mathbf{e}-\mathbf{n},\,\mathbf{r})}\,d\tau, \quad \ldots \quad \ldots \quad (14)$$

and

$$\psi_1 = \frac{1}{4\pi\hbar c}\cdot\frac{e^{\frac{ip}{\hbar}R}}{R}\cdot\int\Big[-\frac{2EV}{\hbar c}+\frac{\partial V}{\partial x_k}\Big\{(\beta_k\beta_4-\beta_4\beta_k)+\frac{E}{mc^2}\beta_k\beta_4^2$$

$$+\frac{ip}{mc}\beta_4(\boldsymbol{\beta}\mathbf{n})\beta_k\Big\}\Big].\,u^+(\mathbf{p})\,e^{\frac{ip}{\hbar}(\mathbf{e}-\mathbf{n},\,\mathbf{r})}\,d\tau. \quad \ldots \quad (15)$$

Again integrating by parts and neglecting the surface integrals, we get

$$\psi_1 = \frac{1}{4\pi\hbar^2c^2}\cdot\frac{e^{\frac{ip}{\hbar}R}}{R}\cdot\Big[-2E+icp\,(\boldsymbol{\beta}\beta_4-\beta_4\boldsymbol{\beta},\,\mathbf{n}-\mathbf{e})+\frac{iEp}{mc}(\boldsymbol{\beta},\,\mathbf{n}-\mathbf{e})\,\beta_4^2$$

$$-\frac{p^2}{m}\beta_4\,(\boldsymbol{\beta}\mathbf{n})\,(\boldsymbol{\beta},\,\mathbf{n}-\mathbf{e})\Big].\,u^+(\mathbf{p})\int V\,e^{\frac{ip}{\hbar}(\mathbf{e}-\mathbf{n},\,\mathbf{r})}\,d\tau. \quad \ldots \quad (16)$$

On the application of the equation of condition (9), (16) then assumes the form

$$\psi_1 = \frac{1}{4\pi\hbar^2c^2}\cdot\frac{e^{\frac{ip}{\hbar}R}}{R}\cdot\Big[-2E\beta_4^2+\frac{2iEp}{mc}(\boldsymbol{\beta}\mathbf{e})\,\beta_4^2+icp\,(\boldsymbol{\beta},\,\mathbf{n}-\mathbf{e})\,\beta_4$$

$$-\frac{p^2}{m}\beta_4\,(\boldsymbol{\beta},\,\mathbf{n}-\mathbf{e})\,(\boldsymbol{\beta}\mathbf{e})\,\beta_4^2+\frac{iEp}{mc}(\boldsymbol{\beta},\,\mathbf{n}-\mathbf{e})\,\beta_4^2-\frac{p^2}{m}\beta_4\,(\boldsymbol{\beta}\mathbf{n})\,(\boldsymbol{\beta},\,\mathbf{n}-\mathbf{e})\,\beta_4^2\Big]$$

$$\times\,u^+(\mathbf{p})\int V\,e^{\frac{ip}{\hbar}(\mathbf{e}-\mathbf{n},\,\mathbf{r})}\,d\tau. \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

The complex conjugate expression is

$$\psi_1^* = \frac{1}{4\pi\hbar^2c^2}\cdot\frac{e^{-\frac{ip}{\hbar}R}}{R}\cdot u^+(\mathbf{p})^*\Big[-2E\beta_4^2 - \frac{2iEp}{mc}\beta_4^2(\boldsymbol{\beta}\mathbf{e}) - icp\,\beta_4(\boldsymbol{\beta}, \mathbf{n}-\mathbf{e})$$
$$-\frac{p^2}{m}\beta_4^2(\boldsymbol{\beta}\mathbf{e})(\boldsymbol{\beta}, \mathbf{n}-\mathbf{e})\beta_4 - \frac{iEp}{mc}\beta_4^2(\boldsymbol{\beta}, \mathbf{n}-\mathbf{e}) - \frac{p^2}{m}\beta_4^2(\boldsymbol{\beta}, \mathbf{n}-\mathbf{e})(\boldsymbol{\beta}\mathbf{n})\beta_4\Big]$$
$$\times\Big(\int Ve^{\frac{ip}{\hbar}(\mathbf{e}-\mathbf{n},\,\mathbf{r})}d\tau\Big)^*. \quad \ldots \quad \ldots \quad \ldots \quad (17')$$

The intensity of the scattered meson, that is, the number of secondary meson that crosses in the direction $\mathbf{n}$ per unit area at a distance R from the scattering centre, is given by the relation

$$I = c\psi_1^\dagger(\boldsymbol{\beta}\mathbf{n})\,\psi_1 \rightarrow v\psi_1^*\beta_4\psi_1, \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

and by (17) and (17′) we get

$$I = \frac{v}{R^2}\cdot\frac{1}{(4\pi\hbar^2c^2)^2}\cdot u^+(\mathbf{p})^*\beta_4^2\Big[4E^2\beta_4 + \frac{4Ep^2}{m}\Big\{(\boldsymbol{\beta}\mathbf{n})^2 - (\boldsymbol{\beta}\mathbf{e})^2\Big\}$$
$$-\frac{p^4}{m^2}\Big\{(\boldsymbol{\beta}\mathbf{e})^2\beta_4(\boldsymbol{\beta}\mathbf{n})(\boldsymbol{\beta}, \mathbf{n}-\mathbf{e}) + (\boldsymbol{\beta}, \mathbf{n}-\mathbf{e})(\boldsymbol{\beta}\mathbf{n})\beta_4(\boldsymbol{\beta}\mathbf{e})^2\Big\}\Big]\beta_4^2 u^+(\mathbf{p})$$
$$\times\Big|\int Ve^{\frac{ip}{\hbar}(\mathbf{e}-\mathbf{n},\,\mathbf{r})}d\tau\Big|^2. \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

When β's are 10-rowed matrices the meson has spin one and for a given $\mathbf{p}$ it has three polarisation states. Hence we shall have to average over the three directions of polarisation in the unperturbed state. To perform this we sum over the three directions of polarisation and divide by 3. The detail method of this calculation and of subsequent evaluation of spurs are given by Booth and Wilson. After performing the above operations we get for meson of spin one

$$I = \frac{v}{R^2}\cdot\frac{1}{(4\pi\hbar^2c^2)^2}\cdot\frac{1}{6E}\,\mathrm{spur}\,\beta_4\Big[4E^2\beta_4 + \frac{4Ep^2}{m}\Big\{(\boldsymbol{\beta}\mathbf{n})^2 - (\boldsymbol{\beta}\mathbf{e})^2\Big\}$$
$$-\frac{p^4}{m^2}\Big\{(\boldsymbol{\beta}\mathbf{e})^2\beta_4(\boldsymbol{\beta}\mathbf{n})(\boldsymbol{\beta}, \mathbf{n}-\mathbf{e}) + (\boldsymbol{\beta}, \mathbf{n}-\mathbf{e})(\boldsymbol{\beta}\mathbf{n})\beta_4(\boldsymbol{\beta}\mathbf{e})^2\Big\}\Big]$$
$$\times\beta_4\Big\{mc^2 + \frac{p^2}{m}(\boldsymbol{\beta}\mathbf{e})^2 + E\beta_4\Big\}\Big|\int Ve^{\frac{ip}{\hbar}(\mathbf{e}-\mathbf{n},\,\mathbf{r})}d\tau\Big|^2. \quad \ldots \quad (20)$$

When β's are 5-rowed matrices the meson has spin zero and it has only one polarisation, viz. in the direction of motion. The equation (20) is valid even in this case with the omission of the factor $\frac{1}{3}$ which arises because of the averaging over the three polarisations.

After evaluating the spurs, we have, for meson of spin one

$$I = \frac{v}{R^2} \cdot \frac{E^2}{(2\pi\hbar^2c^2)^2} \cdot \left[1+\frac{p^4}{6m^2E^2}\left\{1-(\mathbf{en})^2\right\}\right] \left| \int V e^{\frac{ip}{\hbar}(\mathbf{e}-\mathbf{n},\, \mathbf{r})} d\tau \right|^2, \quad .. \quad (21a)$$

and for meson of spin zero

$$I = \frac{v}{R^2} \cdot \frac{E^2}{(2\pi\hbar^2c^2)^2} \cdot \left| \int V e^{\frac{ip}{\hbar}(\mathbf{e}-\mathbf{n},\, \mathbf{r})} d\tau \right|^2. \quad .. \quad .. \quad (21b)$$

Let us now denote the angle between $\mathbf{n}$ and $\mathbf{e}$ by θ and since

$$E = \gamma mc^2,\ p = \gamma mv,\ \gamma = \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}},$$

the differential cross-section $d\phi$ of our scattering process within a solid angle $d\Omega$ is given for meson of spin one by

$$d\phi = \frac{R^2 I}{v} d\Omega = \frac{m^2\gamma^2}{(2\pi\hbar^2)^2}\left\{1+\frac{(\gamma^2-1)^2}{6\gamma^2}\sin^2\theta\right\} \left| \int V e^{\frac{ip}{\hbar}(\mathbf{e}-\mathbf{n},\, \mathbf{r})} d\tau \right|^2 d\Omega \quad .. \quad (22a)$$

and for meson of spin zero by

$$d\phi = \frac{m^2\gamma^2}{(2\pi\hbar^2)^2} \left| \int V e^{\frac{ip}{\hbar}(\mathbf{e}-\mathbf{n},\, \mathbf{r})} d\tau \right|^2 d\Omega. \quad .. \quad .. \quad (22b)$$

For a pure Coulomb field, $V = \frac{ze^2}{r}$, and we have

$$\int V e^{\frac{ip}{\hbar}(\mathbf{e}-\mathbf{n},\, \mathbf{r})} d\tau = \frac{4\pi ze^2\hbar^2}{p^2\,|\mathbf{e}-\mathbf{n}|^2} = \frac{\pi ze^2\hbar^2}{\gamma^2 m^2 v^2 \sin^2\frac{\theta}{2}}.$$

Then (22a) reduces to

$$d\phi = \left(\frac{ze^2}{2mv^2}\right)^2 \cdot \frac{1}{\gamma^2\sin^4\frac{\theta}{2}} \cdot \left\{1+\frac{(\gamma^2-1)^2}{6\gamma^2}\sin^2\theta\right\} d\Omega, \quad .. \quad (23a)$$

and (22b) to

$$d\phi = \left(\frac{ze^2}{2mv^2}\right)^2 \cdot \frac{1}{\gamma^2\sin^4\frac{\theta}{2}} \cdot d\Omega. \quad .. \quad .. \quad .. \quad (23b)$$

The result (23a) agrees with the expressions obtained by Laporte, Massey and Corben, and Majumdar and Gupta; and (23b) is the well-known result as obtained from Klein-Gordon equation.

References.

Booth, F. and Wilson, A. H. (1940).—Radiative Processes involving Fast Mesons. *Proc. Roy. Soc.* (A), **175**, 483.

Christy, R. F. and Kusaka, S. (1941).—The Interaction of γ-rays with Mesotrons. *Phys. Rev.*, **59**, 414.

Duffin, R. J. (1938).—On the Characteristic Matrices of Covariant Systems. *Phys. Rev.*, **54**, 1114.

Kemmer, N. (1939).—The Particle Aspects of Meson. *Proc. Roy. Soc.* (A), **173**, 91.

Laporte, O. (1938).—Elastic Scattering of Yukawa Particles. *Phys. Rev.*, **54**, 905.

Majumdar, R. C. and Gupta, S. (1939–41).—On the Elastic scattering of Meson by a Coulomb Field. *Trans. Bose Research Inst.*, **14**, 183.

Massey, S. W. and Corben, H. C. (1939).—Elastic Collisions of Mesons with Electrons and Protons. *Proc. Camb. Phil. Soc.*, **35**, 463.

Mott, N. F. (1929).—The Scattering of Fast Electrons by Atomic Nuclei. *Proc. Roy. Soc.* (A), **124**, 425.

Sauter, F. (1933).—Zur Stationären Behandlung der elastischen Streuung sehr schneller Elektronen. *Zs. f. Phys.*, **86**, 818.

Wilson, A. H. (1940).—The Calculations of Processes involving Mesons by Matrix Methods. *Proc. Camb. Phil. Soc.*, **36**, 363.