PRESSURE IONISATION AND MAXIMUM RADIUS FOR A COLD BODY.

By P. L. Bhatnagar *and* D. S. Kothari, *University of Delhi.*

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Abstract.

The degree of ionisation in degenerate matter depends essentially on the pressure or density of the material. The results on ionisation of heavy ions (produced in Uranium fission) can be readily utilised to determine the degree of ionisation in degenerate matter. This is done in §1 of the present paper. The results of §1 are applied in §2 to determine the mass-radius relation for cold stellar bodies.

§1. In the case of cold * matter the pressure of the degenerate electron gas is given by the relation

$$\left.\begin{aligned} p &= K\frac{\rho^{\frac{5}{3}}}{\mu^{\frac{5}{3}}}, \\ K &= \frac{8\pi h^2}{15m}\left(\frac{3}{8\pi m_H}\right)^{\frac{5}{3}} \end{aligned}\right\}, \qquad \ldots \quad \ldots \quad \ldots \quad (1)$$

where k is called the degenerate gas constant, ρ is the density of the material, and μ the mean molecular weight per free electron. m represents the mass of electron and m_H that of proton, and h is the Planck's constant.

The mean molecular weight per free electron is the measure of the degree of ionisation of the material: the free electron concentration n being given by

$$n = \frac{\rho}{\mu m_H}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

If the cold body be supposed to consist of material of atomic weight A and atomic number Z, then, for complete ionisation of the material into free electrons and bare nucleii, μ will be equal to $\frac{A}{Z} \simeq 2$ (except for hydrogen when it is equal to 1). If the ionisation be incomplete so that there are only r *free* electrons per atom, then

$$\mu = \frac{A}{r}.$$

The dependence of μ on the density of the material has been worked out previously (Kothari, 1936, 1938) though the theory is admittedly crude being based on several simplifying assumptions.

* Matter will be referred to as *cold* when any *free* electrons present in it constitute a degenerate gas.

Following the recent discovery of Uranium fission, the problem of the energy loss of heavy ions and their degree of ionisation has (following Bohr, 1941) received considerable attention. The degree of ionisation or the average charge of a heavy ion depends primarily on its velocity V; in fact the velocity V_e of the outermost electron in the ion is (almost) equal to the velocity V of the ion. The results on the ionisation of heavy ions can be readily applied to estimate the degree of ionisation in degenerate matter. In the case of ions produced by fission the velocity of the ion is very high compared to the thermal velocity of the molecules of the atmosphere surrounding the ion. It is just the reverse in the case of degenerate matter where, because of its larger mass, the ionic velocity is negligible compared to the velocity of the surrounding free electrons, and the maximum relative velocity of the ion with respect to the free electrons will be the maximum electron-velocity for Fermi-distribution,

$$V = \frac{h}{m}\left(\frac{3n}{8\pi}\right)^{\frac{1}{3}},$$

$$\text{or} \quad V = \frac{h}{m}\left(\frac{3\rho}{8\pi\mu m_H}\right)^{\frac{1}{3}}. \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

The degree of ionisation of the material will be such that the velocity V given by (3) equals the velocity V_e of the outermost bound-electron in the ion.

Knipp and Teller (1941) have calculated on the Thomas-Fermi model the velocity of the most loosely bound electron for different values of the ionic charge.* The first column in the following table gives $\frac{r}{Z}$, and the corresponding values of $\left(\frac{hV_e}{2\pi e^2 Z^{\frac{2}{3}}}\right)$, taken from Knipp and Teller, are given in the second column. The other columns in the table give $\mu = \frac{A}{r}$, density and pressure of the material. The density is determined as follows:—

Assuming $V = V_e$, we have from (3)

$$\rho = \alpha \frac{\left(\frac{hV_e}{2\pi e^2 Z^{\frac{2}{3}}}\right)^3}{\left(\frac{r}{Z}\right)}, \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

where

$$\alpha = \frac{64}{3} A Z m_H \pi^4 \left(\frac{e}{h}\right)^6 m^3,$$

and hence ρ is found by using the values given in columns (1) and (2) of the table. The relation between pressure, density, and μ is expressed by (1).

* The expression for pressure-ionisation given by Kothari (Kothari, 1936, 1938) is based on the assumption of a uniform electron-distribution within the atomic cell.

The values of μ, density, and pressure given in the table refer to the case of iron ($Z = 26$).

TABLE I.

$\frac{r}{Z}$	$\frac{hV_e}{2\pi e^2 Z^{\frac{2}{3}}}$	$A = 55{\cdot}8$; $Z = 26$		
		μ	ρ (gm./cm.^{3})	p (dyne/cm.^{2})
0·01	0·11	215	73·3	$1{\cdot}65 \times 10^{12}$
0·02	0·15	107	92·9	$7{\cdot}78 \times 10^{12}$
0·05	0·21	42·9	102	$4{\cdot}18 \times 10^{13}$
0·10	0·28	21·5	121	$1{\cdot}77 \times 10^{14}$
0·121	0·295	17·7	117	$5{\cdot}56 \times 10^{14}$
0·223	0·411	9·63	171	$1{\cdot}20 \times 10^{15}$
0·522	0·75	4·11	445	$2{\cdot}43 \times 10^{16}$
0·761	1·19	2·82	$1{\cdot}22 \times 10^{3}$	$2{\cdot}44 \times 10^{17}$
0·949	2·29	2·26	$6{\cdot}97 \times 10^{3}$	$6{\cdot}45 \times 10^{18}$

When the ionisation has proceeded to the stage $r = Z-1$, then the orbital velocity of the last bound electron will be

$$V_e = \frac{2\pi e^2 Z}{h},$$

and hence if the ionic velocity exceeds the above value the atom will be completely ionised. It follows, therefore, that for *complete ionisation* of the material

$$V = \frac{h}{m}\left(\frac{3n}{8\pi}\right)^{\frac{1}{3}} \geq \frac{2\pi e^2 Z}{h}$$

or

$$\rho \geq \frac{64}{3} \frac{\pi^4 e^6 Z^3 m^3 \mu m_H}{h^6}.$$

Substituting numerical values and taking the case of iron $\left(\mu = \frac{A}{Z} = 2{\cdot}147\right)$, for complete ionisation the density must exceed

$$1{\cdot}43 \times 10^4 \text{ gm./cm.}^3$$

In fig. 1 the continuous curve represents $\log_{10} p$ against $\log_{10} \rho$ and the ladder marked on the curve denotes the corresponding values of μ. The dotted curve depicts the results previously obtained by Kothari (1938) and are

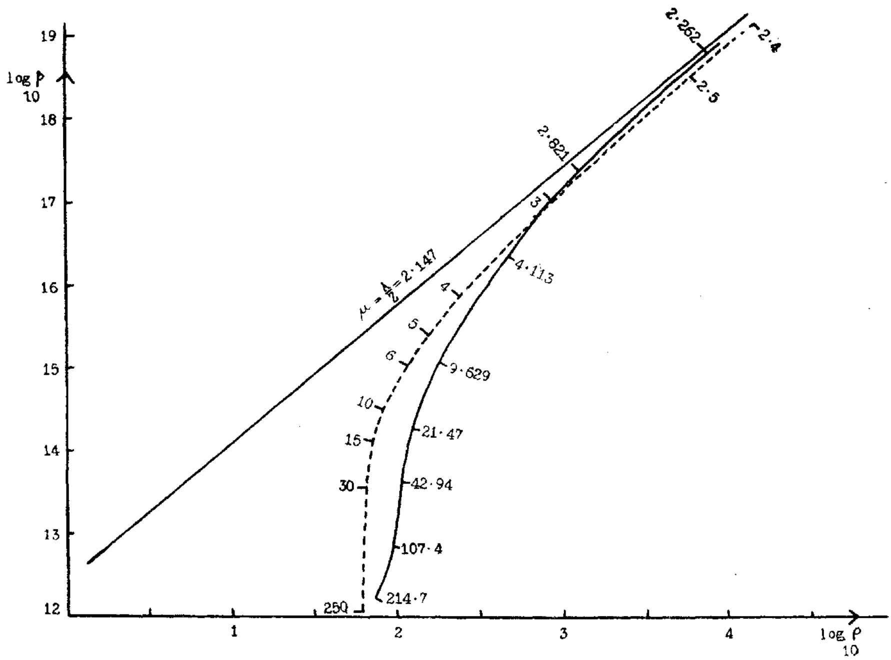

FIG. 1. The figure shows the relation between pressure and density of the material which for numerical evaluation is assumed to be iron. The continuous curve exhibits the results of the present paper. The dotted curve, based on Kothari's earlier calculations, is inserted for comparison. A ladder of μ values has been marked along the curves.

The straight line in the figure would represent the pressure-density relation if the ionisation did not vary with density but was complete $\left(\mu = \frac{A}{Z}\right)$ at all densities.

based, as already remarked, on a uniform electronic distribution within the atomic cell. According to Kothari μ is given by

$$\mu = \frac{\frac{A}{Z}}{\left[1-\left(\frac{\Delta ZA}{\rho}\right)^{\frac{1}{3}}\right]^{\frac{3}{5}}}, \qquad \ldots \qquad \ldots \qquad \ldots \qquad (5)$$

where

$$\Delta = \left[3^{\frac{1}{3}} \, . \, 2\pi^{\frac{2}{3}} \frac{me^2}{h^2} m_H^{\frac{1}{3}}\right]^3 .$$

The values of μ, ρ, and p given in Table II and the dotted curve in fig. 1 are obtained by using this expression for μ in (1). As is to be expected the continuous and the dotted curves are in reasonable agreement.

The straight line in figure 1 represents $\log_{10} p$ against $\log_{10} \rho$ when μ is assumed to be constant and equal to $\frac{A}{Z}$. The straight line, therefore, represents the relation between p and ρ when we ignore the phenomenon of pressure-ionisation whereas the curves take account of it.

TABLE II.

$Z = 26$

μ	ρ (gm./cm.^{3})	p (dyne/cm.^{2})
250	63·0	$9{\cdot}96 \times 10^{11}$
40	64·2	$3{\cdot}52 \times 10^{13}$
30	65·1	$3{\cdot}60 \times 10^{13}$
20	67·5	$7{\cdot}53 \times 10^{13}$
15	70·7	$1{\cdot}31 \times 10^{14}$
10	79·8	$3{\cdot}15 \times 10^{14}$
6	114	$1{\cdot}06 \times 10^{15}$
5	146	$2{\cdot}73 \times 10^{15}$
3	805	$1{\cdot}11 \times 10^{17}$
2·147	∞	∞

§ 2. We shall now apply the above results to determine the maximum radius for a cold body.

In case of a stellar body composed of degenerate matter we have to deal with two opposing tendencies: (i) the mutual gravitational forces tending to diminish the radius, and (ii) the pressure of the degenerate electron gas (varying roughly as the square of the concentration) tending to inflate the configuration. For a spherical aggregate composed of degenerate matter the mass M and the radius R are connected by the well-known relation due to Milne (1932),

$$R = \frac{l}{\mu^{\frac{5}{3}}}\left(\frac{\oplus}{M}\right)^{\frac{1}{3}}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

where

$$l = \frac{5\left(\overset{0}{\omega}_{\frac{3}{2}}\right)^{\frac{1}{3}} K}{2^{\frac{7}{3}}\pi^{\frac{2}{3}}\, G \oplus^{\frac{1}{3}}} = 1{\cdot}932 \times 10^{11} \text{ cm.},$$

$\overset{0}{\omega}_{\frac{3}{2}}$ ($= 132{\cdot}39$) is a constant characteristic of the Emden-solution of Emden's equation of index 3/2, G is the constant of gravitation, and $\oplus$ is the mass of the earth. μ is assumed to be constant throughout the configuration but its dependence on M can be determined as follows:—

The mean density ρ_m of the configuration is

$$\rho_m = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3\mu^5 M^2}{4\pi \oplus l^3}. \qquad \ldots \quad \ldots \quad \ldots \quad (7)$$

For a given value of the density we can find from Table I the corresponding value of μ, and hence the value of M is obtained from (7). Using this value of M in (6) we find the corresponding value of R.

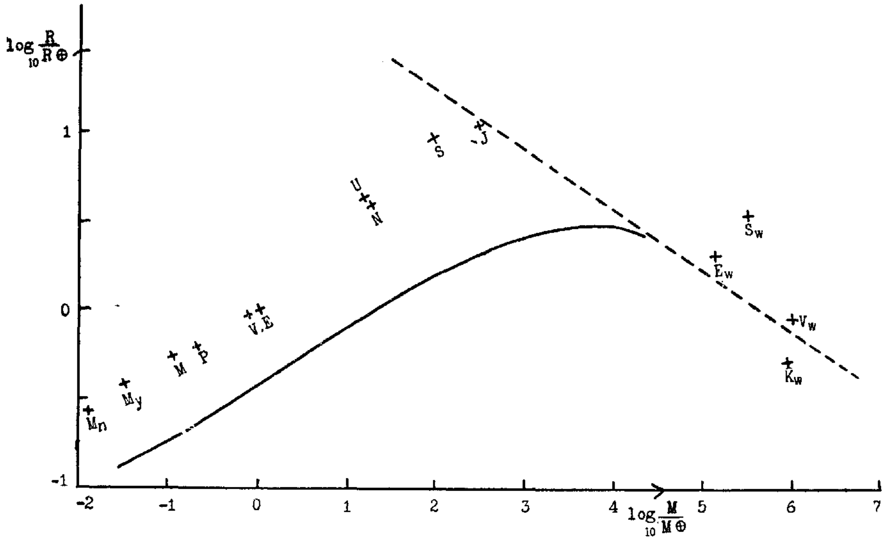

FIG. 2. The figure shows the theoretical mass-radius relation for a cold body. The material is assumed to be iron. The straight line depicts (M-R) relation on the erroneous assumption that the material is always completely ionised. The curve takes account of the dependence of ionisation on density (or mass) in accordance with the theory of pressure ionisation discussed in the text.

M_n = Moon	J = Jupiter	E_w = O_2 Eridani B.
M_y = Mercury	S = Saturn	S_w = Sirius B.
V = Venus	U = Uranus	K_w = A.C. 70°8247.
E = Earth	N = Neptune	V_w = Van Maanen's Star.
M = Mars	P = Pluto	

Fig. 2 exhibits the results of our calculations (the material is assumed to be iron). The observed (M, R) values for planets and white dwarf stars are shown in the figure and, as has been noticed before (Kothari (1938)), the run of the theoretical curve follows the trend of the observed values:—the maximum radius predicted by the theory is about one-third that of Jupiter.*

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* In the calculations given here the material has been assumed to be iron. On the hypothesis of hydrogen abundance the predicted maximum value for the radius of a cold body comes out to be somewhat larger than Jupiter (Kothari 1938).