POLARISATION OF THE VACUUM IN THE MESON THEORY.

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Abstract.

Expressions, correct to the first order in e^2, are obtained for the charges induced in the vacuum of the vector and scalar meson theories by an electric field which is assumed to be much smaller than a critical field. The method of calculation that has been followed is somewhat similar to the subtraction formalism of the positron theory. The deviations due to the existence of the pclarisation of the vacuum from Coulomb's law for the mutual potential energy of point charges have been calculated. The effect of these deviations in modifying the Coulomb scattering law for heavy particles has also been discussed.

Introduction.

It has been shown by various authors (Dirac, 1934; Heisenberg, 1934; Furry and Oppenheimer, 1934; Pierls, 1934; Serber, 1935; Uehling, 1935; Weisskopf, 1936) that the theory of positron produces certain modifications in the Maxwell field equations by the addition of terms which may be linear and non-linear. These modifications occur as the electromagnetic field induces a charge and current distributions due to the creation and annihilation of electron-positron pairs. The presence of the non-linear terms invalidates the principle of superposition of two fields, and gives rise to some processes which are impossible in principle according to ordinary electrodynamics, viz. the process of scattering of light by light, etc. The linear terms, on the other hand, give a non-vanishing polarisability of the vacuum.

The vacuum of the meson theory has also analogous properties. In this theory the existence of positively and negatively charged particles, and the creation and annihilation of meson pairs are the natural consequences of the theory without any special assumption about the vacuum as in the positron theory. But the general problem here is much more complicated than that in the positron theory. In this note, for the sake of simplicity, we wish to investigate the effect which modifies the Maxwell field equations by the addition of linear terms, that is, which corresponds to a non-vanishing polarisability of the vacuum in the meson theory. Furthermore we shall restrict ourselves to the case which is produced by a rapidly varying external electric field. For this purpose we shall start with the Duffin-Kemmer's (1939) formulation of the meson theory; the advantage of this formulation is that the results for theories of particles with spin one and zero are obtained from a single scheme.

As in the positron theory we shall assume that the critical field strength of the meson theory is $F_c = \dfrac{m^2c^3}{\hbar e}$ and restrict ourselves in those regions of space in which the external field intensity is small compared to the critical field, but with no restrictions on the variation of this intensity. Further this field is to be considered switched on adiabatically increasing from zero to some constant value corresponding to the presence of the external charges. On these assumptions an expression for the induced charge distribution will be deduced, defining this quantity as the difference between the contributions to the expectation value of the vacuum distribution in the presence and absence of an external electric field. This expression, however, does not turn out to be finite and cannot give any sensible interpretation to the problem. A quite similar situation also arose in the positron theory, where Heisenberg following the suggestion of Dirac was able to obtain unconditionally convergent expressions for the expectation values of the physical quantities; this convergence is achieved by taking the density matrix in non-diagonal form and subtraction from it of all the singular terms. It is of course possible also to subtract finite terms. Heisenberg has included such a term and his definition of the physical quantities is, of course, consistent with the vanishing of the polarisation of the vacuum for slowly varying and weak fields. In this paper following the method of Pauli and Rose (1936) a similar subtraction formalism will also be adopted to ignore the singular terms in the induced charged distribution and to obtain finite deviation which can be given an unambiguous meaning. We shall not discuss here the question of uniqueness of the result, nor we shall claim it.

1. Equations of Meson and Perturbation Theory.

The meson is defined in the presence of the electromagnetic field by the wave equation

$$\left(\frac{\partial}{\partial x_\mu} - \frac{ie}{\hbar c} A_\mu\right) \beta_\mu \psi + \kappa\psi = 0 \qquad (1)$$

where $\kappa = \frac{mc}{\hbar}$, $x_4 = ict$. The β-operators satisfy the commutation rules

$$\beta_\mu \beta_\nu \beta_\rho + \beta_\rho \beta_\nu \beta_\mu = \beta_\mu \delta_{\nu\rho} + \beta_\rho \delta_{\nu\mu}. \qquad (2)$$

The equation of supplementary condition is given by

$$\left(\frac{\partial}{\partial x_k} - \frac{ie}{\hbar c} A_k\right) \beta_k \beta_4^2 \psi + \kappa(1-\beta_4^2)\psi = 0 \qquad (3)$$

with the help of which we can eliminate those components of ψ for which $\beta_4\psi$ is zero and which are not directly quantised. Repeated suffixes mean summation; Greek suffixes run from 1 to 4 and Latin from 1 to 3.

The plane wave solution of the equation of free meson is

$$\phi_m = \frac{1}{\sqrt{V}} u_m . \exp.\{-i\varepsilon_m E_m t/\hbar + i(\mathbf{p}_m \mathbf{r})/\hbar\} \qquad (4)$$

where $\varepsilon_m = \pm 1$, $E_m = c\,(p_m^2 + m^2c^2)^{\frac{1}{2}}$. In the vector meson theory (spin one), the β's are ten-row square matrices and u's are column matrices of ten elements. For a given $\mathbf{p}$ there are six linearly independent solutions of the form (4), three of which are associated with the positively charged states for which $\varepsilon_m = +1$ and three with the negatively charged states for which $\varepsilon_m = -1$, and correspond to the three states of polarisation which are taken to be transverse and longitudinal to the direction of motion. In the scalar meson theory (spin zero), the β's are five-row square matrices; there are two linearly independent solutions, one of which is associated with the positively charged state and the other with the negative one, and the polarisation is longitudinal.

We suppose that the meson to be initially in the state ϕ_n. At any subsequent time after the introduction of the field let the wave function be ψ_n which is expanded as

$$\psi_n = \sum_m c_{nm} \phi_m. \qquad (5)$$

We now write the equation (1) in the form

$$\left(\beta_\mu \frac{\partial}{\partial x_\mu} + \kappa\right)\psi = \frac{ie}{\hbar c} A_\mu \beta_\mu \psi,$$

and eliminate with the help of (3) from the right hand side those components of ψ for which $\beta_4\psi$ is zero, and get

$$\left(\beta_\mu \frac{\partial}{\partial x_\mu}+\kappa\right)\psi = \frac{ie}{\hbar c} A_\mu\beta_\mu\left\{1-\frac{1}{\kappa}\beta_k\frac{\partial}{\partial x_k}\right\}\beta_4^2\psi - \frac{e^2}{\hbar^2c^2\kappa}(\mathbf{A}\beta)^2\beta_4^2\psi. \qquad (6)$$

Now since

$$\left(\beta_\mu \frac{\partial}{\partial x_\mu}+\kappa\right)\phi_m = 0, \text{ and } \beta_k\beta_4^2\frac{\partial\phi_m}{\partial x_k}+\kappa(1-\beta_4^2)\phi_m = 0$$

we have from (5) and (6)

$$\frac{1}{ic}\sum_m \dot{c}_{nm}\beta_4\phi_m = \frac{ie}{\hbar c}\sum_m c_{nm}\left\{A_\mu\beta_\mu+\frac{ie}{\hbar c\kappa}(\mathbf{A}\beta)^2\beta_4^2\right\}\phi_m. \qquad (7)$$

Multiplying by $\phi_r^\dagger$ on the left and integrating over the whole co-ordinates space and applying the normalising condition

$$\frac{1}{i}\int\phi_r^\dagger\beta_4\phi_m d\mathbf{r} = \varepsilon_r\delta_{rm}$$

where $\phi_r^\dagger = i\phi_r^*\eta_4$, $\eta_4 = 2\beta_4^2-1$, we obtain

$$\dot{c}_{nr}(t) = \frac{i\varepsilon_r}{\hbar}\sum_m c_{nm}H_{mr}\exp.\frac{i}{\hbar}(\varepsilon_rE_r-\varepsilon_mE_m)t, \qquad (8)$$

where

$$H_{mr} = \frac{e}{V}\int u_r^\dagger\left\{A_\mu\beta_\mu+\frac{ie}{\hbar c\kappa}(\mathbf{A}\beta)^2\beta_4^2\right\}u_m\,.\exp.\frac{i}{\hbar}(\mathbf{p}_m-\mathbf{p}_r,\mathbf{r})d\mathbf{r}. \qquad (9)$$

For the sake of simplicity we shall work out the case in which $\mathbf{A}=0$ and $A_0\neq 0$ and be interested only in the charge density but not in the current density. Since we shall confine ourselves to the approximation in which effects proportional to the powers of the constant $\alpha = \frac{e^2}{\hbar c}$ higher than the first will be neglected, there will be no loss of generality if we expand A_0 in plane waves of which the one Fourier component is

$$A_0 = a_0\exp.\frac{i}{\hbar}\{(\mathbf{kr})-ck_0t\}+\text{conj.}, \text{ and } \mathbf{A}=0, \qquad (10)$$

where $\mathbf{k}$ is the propagation vector, and according to the Maxwell equations

$$j_0(\mathbf{r}) = \frac{k^2}{4\pi\hbar^2}A_0(\mathbf{r}) \qquad (11)$$

where $j_0(\mathbf{r})$ is the charge density which produces the field A_0.

Then following the method and assumption of Heisenberg (1934), we get in the usual way from (8), (9) and (10)

$$c_{nm}(t) = -e\varepsilon_m\left[u_m(\mathbf{p}_n+\mathbf{k})^*\beta_4u(\mathbf{p}_n)\,.\frac{a_0\exp.\frac{i}{\hbar}(\varepsilon_mE_m-\varepsilon_nE_n-ck_0)t}{\varepsilon_mE_m-\varepsilon_nE_n-ck_0}\right.$$
$$\left.+u_m(\mathbf{p}_n-\mathbf{k})^*\beta_4u(\mathbf{p}_n)\,.\frac{a_0^*\exp.\frac{i}{\hbar}(\varepsilon_mE_m-\varepsilon_nE_n+ck_0)t}{\varepsilon_mE_m-\varepsilon_nE_n+ck_0}\right]+\delta_{nm}.$$

Hence the solution of the equation (1) in the first order perturbation theory when A_μ is given by (10) can be written in the form

$$\psi^\pm(\mathbf{p},\mathbf{r}) = \phi^\pm(\mathbf{p},\mathbf{r})+\phi^\pm(\mathbf{p},\mathbf{r})' \qquad (12)$$

where

$$\phi^{\pm}(\mathbf{p}, \mathbf{r})' = -e\Big[u^{+}(\mathbf{p}+\mathbf{k})^{*}\beta_4 u^{\pm}(\mathbf{p}) . \frac{a_0 \exp. \frac{i}{\hbar}\{E(\mathbf{p}+\mathbf{k}) \mp E(\mathbf{p}) - ck_0\}t}{E(\mathbf{p}+\mathbf{k}) \mp E(\mathbf{p}) - ck_0}\phi^{+}(\mathbf{p}+\mathbf{k}, \mathbf{r})$$

$$+ u^{-}(\mathbf{p}+\mathbf{k})^{*}\beta_4 u^{\pm}(\mathbf{p}) . \frac{a_0 \exp. -\frac{i}{\hbar}\{E(\mathbf{p}+\mathbf{k}) \pm E(\mathbf{p}) + ck_0\}t}{E(\mathbf{p}+\mathbf{k}) \pm E(\mathbf{p}) + ck_0}\phi^{-}(\mathbf{p}+\mathbf{k}, \mathbf{r})$$

$$+ u^{+}(\mathbf{p}-\mathbf{k})^{*}\beta_4 u^{\pm}(\mathbf{p}) . \frac{a_0^{*} \exp. \frac{i}{\hbar}\{E(\mathbf{p}-\mathbf{k}) \mp E(\mathbf{p}) + ck_0\}t}{E(\mathbf{p}-\mathbf{k}) \mp E(\mathbf{p}) + ck_0}\phi^{+}(\mathbf{p}-\mathbf{k}, \mathbf{r})$$

$$+ u^{-}(\mathbf{p}-\mathbf{k})^{*}\beta_4 u^{\pm}(\mathbf{p}) . \frac{a_0^{*} \exp. -\frac{i}{\hbar}\{E(\mathbf{p}-\mathbf{k}) \pm E(\mathbf{p}) - ck_0\}t}{E(\mathbf{p}-\mathbf{k}) \pm E(\mathbf{p}) - ck_0}\phi^{-}(\mathbf{p}-\mathbf{k}, \mathbf{r})\Big] \quad .. \quad (13)$$

where the index + or − corresponds to positively or negatively charged state.

2. The Induced Charge Density.

The charge density in the present theory is defined by

$$\rho = e\psi(\mathbf{r}')^{*}\beta_4\psi(\mathbf{r}'').$$

Following Heisenberg's rule we replace this charge density as in a previous paper of the present author (1943) by

$$\rho = \frac{e}{2}\ \{\psi(\mathbf{r}')^{*}\beta_4\psi(\mathbf{r}'') + \beta_4\psi(\mathbf{r}'').\psi(\mathbf{r}')^{*}\beta_4^2\} \quad .. \quad .. \quad .. \quad (14)$$

where in the second term $\beta_4\psi$ is a single row matrix and $\psi^{*}\beta_4^2$ is a single column matrix, and this definition is taken for symmetricising the density matrix between positively and negatively charged mesons. Now if

$$\psi_0 = \sum_{p}\{a(\mathbf{p})\phi^{+}(\mathbf{p}) + b^{*}(\mathbf{p})\phi^{-}(\mathbf{p})\} \quad .. \quad .. \quad .. \quad (15)$$

is the wave function unperturbed by the field and ψ_1 is the correction in the first order perturbation theory, then

$$\psi_1 = \sum_{p}\{a(\mathbf{p})\phi^{+}(\mathbf{p})' + b^{*}(\mathbf{p})\phi^{-}(\mathbf{p})'\} \quad .. \quad .. \quad .. \quad (16)$$

where $\phi^{+}(\mathbf{p})'$ and $\phi^{-}(\mathbf{p})'$ are given by (13). Hence

$$(\mathbf{r}'\,|\,\rho\,|\,\mathbf{r}'') = (\mathbf{r}'\,|\,\rho_0\,|\,\mathbf{r}'') + (\mathbf{r}'\,|\,\delta\rho\,|\,\mathbf{r}'')$$

where

$$(\mathbf{r}'\,|\,\rho_0\,|\,\mathbf{r}'') = \frac{e}{2}\ \{\psi_0(\mathbf{r}')^{*}\beta_4\psi_0(\mathbf{r}'') + \beta_4\psi_0(\mathbf{r}'').\psi_0(\mathbf{r}')^{*}\beta_4^2\} \quad .. \quad .. \quad (17)$$

and

$$(\mathbf{r}'\,|\,\delta\rho\,|\,\mathbf{r}'') = \frac{e}{2}\{\psi_0(\mathbf{r}')^{*}\beta_4\psi_1(\mathbf{r}'') + \beta_4\psi_0(\mathbf{r}'').\psi_1(\mathbf{r}')^{*}\beta_4^2$$
$$+ \psi_1(\mathbf{r}')^{*}\beta_4\psi_0(\mathbf{r}'') + \beta_4\psi_1(\mathbf{r}'').\psi_0(\mathbf{r}')^{*}\beta_4^2\} \quad .. \quad .. \quad (18)$$

to a first approximation. If (15) and (16) are now substituted in (17) and (18) and only the diagonal elements with respect to time variable are retained, $(\mathbf{r}'|\rho_0|\mathbf{r}'')$ for the vacuum vanishes while (18) in non-diagonal form in space co-ordinates becomes

$$(\mathbf{r}'|\delta\rho|\mathbf{r}'') = \frac{e}{2}\sum_{p}\Big[(2N^+(\mathbf{p})+1)\{\phi^+(\mathbf{p},\mathbf{r}')^*\beta_4\phi^+(\mathbf{p},\mathbf{r}'')'+\phi^+(\mathbf{p},\mathbf{r}')'^*\beta_4\phi^+(\mathbf{p},\mathbf{r}'')\}$$
$$+(2N^-(\mathbf{p})+1)\{\phi^-(\mathbf{p},\mathbf{r}')^*\beta_4\phi^-(\mathbf{p},\mathbf{r}'')'+\phi^-(\mathbf{p},\mathbf{r}')'^*\beta_4\phi^-(\mathbf{p},\mathbf{r}'')\}\Big]$$

where $N^+(\mathbf{p})$ and $N^-(\mathbf{p})$ give the number of positive and negative mesons in the state $\mathbf{p}$. For the vacuum $N^+(\mathbf{p}) = N^-(\mathbf{p}) = 0$, and we get

$$(\mathbf{r}'|\delta\rho|\mathbf{r}'') = \frac{e}{2}\sum_{p}\Big[\phi^+(\mathbf{p},\mathbf{r}')^*\beta_4\phi^+(\mathbf{p},\mathbf{r}'')'+\phi^+(\mathbf{p},\mathbf{r}')'^*\beta_4\phi^+(\mathbf{p},\mathbf{r}'')$$
$$+\phi^-(\mathbf{p},\mathbf{r}')^*\beta_4\phi^-(\mathbf{p},\mathbf{r}'')'+\phi^-(\mathbf{p},\mathbf{r}')'^*\beta_4\phi^-(\mathbf{p},\mathbf{r}'')\Big]$$

Then by (13) after some calculations this assumes the form

$$(\mathbf{r}'|\delta\rho|\mathbf{r}'') = -\frac{e^2}{V}\sum_{q}\Big[\frac{\{u^-(\mathbf{q}+\mathbf{k}/2)^*\beta_4u^+(\mathbf{q}-\mathbf{k}/2)\}\{u^+(\mathbf{q}-\mathbf{k}/2)^*\beta_4u^-(\mathbf{q}+\mathbf{k}/2)\}}{E(\mathbf{q}+\mathbf{k}/2)+E(\mathbf{q}-\mathbf{k}/2)+ck_0}$$
$$+\frac{\{u^+(\mathbf{q}+\mathbf{k}/2)^*\beta_4u^-(\mathbf{q}-\mathbf{k}/2)\}\{u^-(\mathbf{q}-\mathbf{k}/2)^*\beta_4u^+(\mathbf{q}+\mathbf{k}/2)\}}{E(\mathbf{q}+\mathbf{k}/2)+E(\mathbf{q}-\mathbf{k}/2)-ck_0}\Big]A_0(\mathbf{R})\ \exp.\frac{i}{\hbar}(\mathbf{qr}) \quad .. \quad (19)$$

where

$$\mathbf{R} = \frac{\mathbf{r}'+\mathbf{r}''}{2}, \quad \mathbf{r} = \mathbf{r}''-\mathbf{r}'. \quad .. \quad .. \quad .. \quad .. \quad (20)$$

We shall now sum over the directions of polarisation in the states $\mathbf{q}+\mathbf{k}/2$ and $\mathbf{q}-\mathbf{k}/2$, the detail method of which and of the subsequent spur calculations are given by Booth and Wilson (1940). On performing the above operations and replacing the summation over q by integration, (19) assumes the form[a]

$$(\mathbf{r}'|\delta\rho^0|\mathbf{r}'') = -\frac{e^2}{2c}\cdot\frac{A_0(\mathbf{R})}{(2\pi\hbar)^3}\int\frac{(\epsilon+\epsilon')(\epsilon-\epsilon')^2}{\epsilon\epsilon'\{(\epsilon+\epsilon')^2-k_0^2\}}\ \exp.\frac{i}{\hbar}(\mathbf{qr})d\mathbf{q} \quad .. \quad (21a)$$

for the scalar meson theory (spin zero), and

$$(\mathbf{r}'|\delta\rho^1|\mathbf{r}'') = -\frac{3e^2}{2c}\cdot\frac{A_0(\mathbf{R})}{(2\pi\hbar)^3}\int\frac{\epsilon+\epsilon'}{\epsilon\epsilon'\{(\epsilon+\epsilon')^2-k_0^2\}}\Big\{(\epsilon-\epsilon')^2$$
$$+\frac{2}{3m^2c^2}\left(q^2k^2-(\mathbf{qk})^2\right)\Big\}\ \exp.\frac{i}{\hbar}(\mathbf{qr})d\mathbf{q} \quad .. \quad (21b)$$

for the vector meson theory (spin one), where

$$\epsilon = [(\mathbf{q}-\mathbf{k}/2)^2+m^2c^2]^{\frac{1}{2}}, \quad \epsilon' = [(\mathbf{q}+\mathbf{k}/2)^2+m^2c^2]^{\frac{1}{2}}. \quad .. \quad .. \quad (22)$$

For $k_0 = 0$ the expression (21a) for the scalar meson theory is in agreement with the corresponding expression given by Pauli and Weisskopf (1934). The expressions (21a) and (21b) are the differences between the charge density matrix in the presence and the absence of the external field in the approximation in question. They are the Fourier amplitudes

[a] Throughout the paper the results with index 0 refer to the scalar meson theory and with index 1 to the vector meson theory.

of the charge induced by an electric field which is everywhere small compared to the critical field $F_c = \frac{m^2c^3}{\hbar e}$. Using the relation (11), (21*a*) can be written in the form

$$(\mathbf{r}' \,|\, \delta\rho^0 \,|\, \mathbf{r}'') = -\frac{\alpha}{8\pi^2} \cdot \frac{k^2}{\hbar^2} A_0(\mathbf{R}) F^0(\mathbf{k}, k_0, \mathbf{r})$$

$$= -\frac{\alpha}{2\pi} j_0(\mathbf{R}) F^0(\mathbf{k}, k_0, \mathbf{r}) \quad \ldots \quad \ldots \quad \ldots (23a)$$

where

$$F^0(\mathbf{k}, k_0, \mathbf{r}) = \frac{1}{2\pi k^2} \int \frac{(\epsilon+\epsilon')(\epsilon-\epsilon')^2}{\epsilon\epsilon'\{(\epsilon+\epsilon')^2 - k_0^2\}} \exp. \frac{i}{\hbar}(\mathbf{qr})d\mathbf{q}, \quad \ldots \quad \ldots (24a)$$

and (21*b*) in the form

$$(\mathbf{r}' \,|\, \delta\rho^1 \,|\, \mathbf{r}'') = -\frac{3\alpha}{8\pi^2} \cdot \frac{k^2}{\hbar^2} A_0(\mathbf{R}) F^1(\mathbf{k}, k_0, \mathbf{r})$$

$$= -\frac{3\alpha}{2\pi} \cdot j_0(\mathbf{R}) F^1(\mathbf{k}, k_0, \mathbf{r}) \quad \ldots \quad \ldots \quad \ldots (23b)$$

where

$$F^1(\mathbf{k}, k_0, \mathbf{r}) = \frac{1}{2\pi k^2} \int \frac{(\epsilon+\epsilon')}{\epsilon\epsilon'\{(\epsilon+\epsilon')^2 - k_0^2\}} \left\{(\epsilon-\epsilon')^2 + \frac{2}{3m^2c^2}\left(q^2k^2 - (\mathbf{qk})^2\right)\right\} \exp. \frac{i}{\hbar}(\mathbf{qr})d\mathbf{q}.$$

$$\ldots \quad \ldots (24b)$$

For $\mathbf{r} = 0$ the integrals (24*a*) and (24*b*) do not turn out to be finite. Following the method in the positron theory we shall introduce suitably chosen subtractive terms depending on $\mathbf{r}$ in such a way that the difference is finite for $\mathbf{r} = 0$. These subtractive terms will in general be the singular terms. It is of course possible also to subtract the finite terms. These terms are to be so chosen that they normalise the polarisation of the vacuum for slowly varying and weak fields to zero. For this purpose the method of Pauli and Rose is followed here. For large q or what turns out to be the same for small values of k,

$$\epsilon \sim \epsilon' \sim (m^2c^2 + q^2)^{\frac{1}{2}}$$

and to a better approximation

$$(\epsilon - \epsilon')^2 \sim \frac{(\mathbf{qk})^2}{(m^2c^2+q^2)},$$

$$\left[\epsilon\epsilon'(\epsilon+\epsilon')\left\{1 - \frac{k_0^2}{(\epsilon+\epsilon')^2}\right\}\right]^{-1} \sim \frac{1}{2(m^2c^2+q^2)^{\frac{3}{2}}}\left\{1 - \frac{3}{8}\frac{k^2}{m^2c^2+q^2}\right.$$

$$\left. + \frac{5}{8}\frac{(\mathbf{qk})^2}{(m^2c^2+q^2)^2} + \frac{1}{4}\frac{k_0^2}{m^2c^2+q^2}\right\}.$$

So we can split up each of the integrals (24*a*) and (24*b*) into two terms

$$F^0(\mathbf{k}, k_0, \mathbf{r}) = F_1^0(\mathbf{k}, k_0, \mathbf{r}) + F_2^0(\mathbf{k}, k_0, \mathbf{r}), \quad \ldots \quad \ldots \quad \ldots (25a)$$

and

$$F^1(\mathbf{k}, k_0, \mathbf{r}) = F_1^1(\mathbf{k}, k_0, \mathbf{r}) + F_2^1(\mathbf{k}, k_0, \mathbf{r}), \quad \ldots \quad \ldots \quad \ldots (25b)$$

where

$$F_1^0(\mathbf{k}, k_0, \mathbf{r}) = \frac{1}{4\pi} \int \left[\frac{q^2\cos^2\theta}{m^2c^2 +}\right] \frac{\exp. \frac{i}{\hbar}(\mathbf{qr})}{(m^2c^2+q^2)^{\frac{3}{2}}} d\mathbf{q}, \quad \ldots \quad \ldots (26a)$$

and

$$F_1^1(\mathbf{k},\ k_0,\ \mathbf{r}) = \frac{1}{4\pi}\int\left[\frac{q^2\cos^2\theta}{m^2c^2+q^2}+\frac{2q^2\sin^2\theta}{3m^2c^2}\left\{1-\frac{1}{8(m^2c^2+q^2)}\left(3k^2-5k^2\ \cos^2\theta-2k_0^2\right)\right\}\right]$$

$$\times\frac{\exp.\frac{i}{\hbar}(\mathbf{qr})}{(m^2c^2+q^2)^{\frac{3}{2}}}\,d\mathbf{q}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (26b)$$

θ being the angle between $\mathbf{q}$ and $\mathbf{k}$, and where for $r=0$

$$f^0(\mathbf{k},\ k_0) = F_2^0(\mathbf{k},\ k_0,\ 0) = \frac{1}{2\pi k^2}\int\left[\frac{(\epsilon+\epsilon')(\epsilon-\epsilon')^2}{\epsilon\epsilon'\{(\epsilon+\epsilon')^2-k_0^2\}}-\frac{k^2}{2}\cdot\frac{q^2\cos^2\theta}{(m^2c^2+q^2)^{\frac{5}{2}}}\right]d\mathbf{q}, \qquad \ldots \quad (27a)$$

and

$$f^1(\mathbf{k},\ k_0) = F_2^1(\mathbf{k},\ k_0,\ 0) = \frac{1}{2\pi k^2}\int\left[\frac{\epsilon+\epsilon'}{\epsilon\epsilon'\{(\epsilon+\epsilon')^2-k_0^2\}}\left\{(\epsilon-\epsilon')^2+\frac{2k^2q^2\sin^2\theta}{3m^2c^2}\right\}\right.$$

$$\left.-\frac{k^2}{2}\left\{\frac{q^2\cos^2\theta}{m^2c^2+q^2}+\frac{2q^2\sin^2\theta}{3m^2c^2}\left(1-\frac{1}{8(m^2c^2+q^2)}\left(3k^2-5k^2\cos^2\theta-2k_0^2\right)\right)\right\}\frac{1}{(m^2c^2+q^2)^{\frac{3}{2}}}\right]d\mathbf{q}. \quad (27b)$$

For $\mathbf{r}=0$ the integrand in F_1^0 decreases like q^{-3} for large values of q and therefore F_1^0 diverges logarithmically; the terms in the integrand in F_1^1 decrease either as q^{-3} or q^{-1} for large q and therefore F_1^1 diverges both logarithmically and quadratically; while both in F_2^0 and F_2^1 integrands decrease with the higher order q^{-5} so that the total integrals given by (27a) and (27b) are convergent. If one identifies the subtractive terms with $F_1^0(\mathbf{k},\ k_0,\ \mathbf{r})$ and $F_1^1(\mathbf{k},\ k_0,\ \mathbf{r})$ given by (26a) and (26b) respectively, one then gets from (23a) and (23b) by putting the 'off diagonal distance', $\mathbf{r}$, equal to zero

$$\delta j_0^{\cdot 0} = -\frac{\alpha}{2\pi}f^0(\mathbf{k},k_0)j_0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28a)$$

for the scalar meson theory, and

$$\delta j_0^{\cdot 1} = -\frac{3\alpha}{2\pi}f^1(\mathbf{k},k_0)j_0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28b)$$

for the vector meson theory, where the functions $f^0(\mathbf{k},\ k_0)$ and $f^1(\mathbf{k},\ k_0)$ are given by (27a) and (27b) respectively. The expressions (28a) and (28b) may be considered as the induced charge densities in the two theories.

On evaluating the integrals (27a) and (27b) following the procedure of Pauli and Rose, we get

$$f^0(\mathbf{k},k_0) = f(L) = -\frac{L}{3}\int_0^1\frac{z^4dz}{1+L\,(1-z^2)}, \qquad \ldots \quad \ldots \quad (29a)$$

and

$$f^1(\mathbf{k},k_0) = \left(1-\frac{4}{3}L\right)f(L), \qquad \ldots \quad \ldots \quad \ldots \quad (29b)$$

where

$$L = (k^2-k_0^2)/4m^2c^2. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

It should be noticed that both $f^0(\mathbf{k},k_0)$ and $f^1(\mathbf{k},k_0)$ do not depend on k and k_0 separately but on the single combination L which result is a consequence of the Lorentz invariance of the formalism.

$f(L)$ can be written in the form

$$f(L) = -\frac{1}{3}\left[-\frac{4}{3}-\frac{1}{L}+\frac{(L+1)^2}{L}\phi(L)\right] \qquad \text{(31)}$$

where

$$\phi(L) = \int_0^1 \frac{dz}{1+L(1-z^2)}. \qquad \text{(32)}$$

The evaluation of (32) is simple and the results given by Pauli and Rose are as follows:

$$\phi(L) = [L(1+L)]^{-\frac{1}{2}} \log [(1+L)^{\frac{1}{2}}+L^{\frac{1}{2}}], \qquad \text{for } L>0, \qquad (33a)$$

$$\phi(L) = [|L|(1-|L|)]^{-\frac{1}{2}} \sin^{-1}|L|^{\frac{1}{2}}, \qquad \text{for } -1<L<0, \qquad (33b)$$

$$\phi(L) = -[|L|(|L|-1)]^{-\frac{1}{2}} \log [(|L|-1)^{\frac{1}{2}}+|L|^{\frac{1}{2}}] \qquad \text{for } L<-1. \qquad (33c)$$

In the last case the integrand has a singularity at one point of the integration path, viz. at $z=\left(\frac{|L|-1}{|L|}\right)^{\frac{1}{2}}$, and one has there to take the principal value.

For large positive values of L, we get asymptotically

$$f^0(k, k_0) \cong \frac{4}{9}-\frac{1}{3}\log(2L^{\frac{1}{2}}), \qquad (34a)$$

and

$$f^1(k, k_0) \cong \frac{4}{9}L\left[-\frac{4}{3}+\log(2L^{\frac{1}{2}})\right]. \qquad (34b)$$

For $|L| \ll 1$, one get

$$f^0(k, k_0) = f^1(k, k_0) \cong -\frac{1}{15}L. \qquad (35)$$

In the case $k_0=0$, we have $L=k^2/4m^2c^2$ and (29*a*), (29*b*), (31) and (33*a*) lead to the results

$$f^0(k^2) = -\frac{k^2}{3}\int_0^1 \frac{z^4 dz}{4m^2c^2+k^2(1-z^2)}$$

$$= -\frac{1}{3}\left[-\frac{4}{3}-\frac{4m^2c^2}{k^2}+\left(1+\frac{4m^2c^2}{k^2}\right)^{\frac{3}{2}}\log\left\{\left(1+\frac{k^2}{4m^2c^2}\right)^{\frac{1}{2}}+\frac{k}{2mc}\right\}\right], \qquad (36a)$$

and

$$f^1(k^2) = \left(1-\frac{k^2}{3m^2c^2}\right)f^0(k^2). \qquad (36b)$$

When $k^2 \ll m^2c^2$, we get

$$f^0(k^2) = -\frac{1}{60}\frac{k^2}{m^2c^2}+\frac{1}{840}\frac{k^4}{m^4c^4}+\cdots\cdots, \qquad (37a)$$

and

$$f^1(k^2) = -\frac{1}{60}\frac{k^2}{m^2c^2}+\frac{17}{2520}\frac{k^4}{m^4c^4}+\cdots\cdots \qquad (37b)$$

Hence from (28*a*) and (28*b*) one has

$$\delta j_0^0 = -\frac{\alpha}{2\pi}\left\{\frac{1}{60}\left(\frac{\hbar}{mc}\right)^2 \nabla^2 j_0+\frac{1}{840}\left(\frac{\hbar}{mc}\right)^4 \nabla^4 j_0+\cdots\cdot\right\} \qquad (38a)$$

for the scalar meson theory, and

$$\delta j_0^1 = -\frac{3\alpha}{2\pi}\left\{\frac{1}{60}\left(\frac{\hbar}{mc}\right)^2 \nabla^2 j_0+\frac{17}{2520}\left(\frac{\hbar}{mc}\right)^4 \nabla^4 j_0+\cdots\cdot\right\} \qquad (38b)$$

for the vector meson theory.

3. The Function $U(r)$.

In our subsequent discussion we shall restrict ourselves to the time independent A_0 and obtain an expression for the induced charge in a form given by Dr. Uehling in the positron theory. In this case $k_0 = 0$, we have from (23*a*), (23*b*) and (28*a*), (28*b*) for the induced charge density:

$$\delta j_0^0(\mathbf{r}') = -\frac{\alpha}{2\pi}\cdot\frac{k^2}{4\pi\hbar^2}A_0(\mathbf{r}')f^0(k^2) \qquad \text{(39a)}$$

for the scalar meson theory, and

$$\delta j_0^1(\mathbf{r}') = -\frac{3\alpha}{2\pi}\cdot\frac{k^2}{4\pi\hbar^2}A_0(\mathbf{r}')f^1(k^2) \qquad \text{(39b)}$$

for the vector meson theory, where $f^0(k^2)$ and $f^1(k^2)$ are given by (36*a*) and (36*b*) respectively. If now there be a continuous distribution of inducing charge $j_0(\mathbf{r}'')$, we have for the scalar potential

$$A_0(\mathbf{r}') = \int\frac{j_0(\mathbf{r}'')d\mathbf{r}''}{|\mathbf{r}''-\mathbf{r}'|}\ ,\ \int A_0(\mathbf{r}')\exp.-\frac{i}{\hbar}(\mathbf{r}'\mathbf{k})d\mathbf{r}' = \frac{4\pi\hbar^2}{k^2}\int j_0(\mathbf{r}'')\exp.-\frac{i}{\hbar}(\mathbf{r}''\mathbf{k})d\mathbf{r}'', \qquad (40)$$

and for a single Fourier component with wave vector $\mathbf{k}$

$$A_0(\mathbf{r}') = \frac{4\pi\hbar^2}{Vk^2}\int j_0(\mathbf{r}'')\exp.-\frac{i}{\hbar}(\mathbf{rk})d\mathbf{r}'' = -\frac{4\pi\hbar^4}{Vk^4}\int j_0(\mathbf{r}'')\nabla^2\exp.-\frac{i}{\hbar}(\mathbf{rk})d\mathbf{r}''$$

$$= \frac{4\pi\hbar^6}{Vk^6}\int j_0(\mathbf{r}'')\nabla^4\exp.-\frac{i}{\hbar}(\mathbf{rk})d\mathbf{r}'', \qquad (41)$$

where $\mathbf{r} = \mathbf{r}''-\mathbf{r}'$. On partial integration

$$A_0(\mathbf{r}') = -\frac{4\pi\hbar^4}{Vk^4}\int\exp.-\frac{i}{\hbar}(\mathbf{rk}).\nabla^2 j_0(\mathbf{r}'')d\mathbf{r}'' = \frac{4\pi\hbar^6}{Vk^6}\int\exp.-\frac{i}{\hbar}(\mathbf{rk}).\nabla^4 j_0(\mathbf{r}'')d\mathbf{r}''. \qquad (42)$$

Hence combining all the Fourier components, we get from (39*a*)

$$\delta j_0^0(\mathbf{r}') = \frac{\alpha}{2\pi}\cdot\frac{1}{8\pi^3\hbar}\int\int\frac{f^0(k^2)}{k^2}\exp.-\frac{i}{\hbar}(\mathbf{rk}).\nabla^2 j_0(\mathbf{r}'')d\mathbf{r}''d\mathbf{k}$$

$$= \frac{\alpha}{8\pi^2}\int U^0(r)\nabla^2 j_0(\mathbf{r}'')d\mathbf{r}'' \qquad \text{(43a)}$$

for the scalar meson theory, where

$$U^0(r) = \frac{1}{2\pi^2\hbar}\int\frac{f^0(k^2)}{k^2}\exp.-\frac{i}{\hbar}(\mathbf{rk})d\mathbf{k} = \frac{2}{\pi}\int\frac{f^0(k^2)}{k^2}\cdot\frac{1}{r}\sin\left(\frac{kr}{\hbar}\right)kdk, \qquad \text{(44a)}$$

and from (39*b*), (36*b*) and (42)

$$\delta j_0^1(\mathbf{r}') = \frac{3\alpha}{8\pi^2}\int U^0(r)\left[\nabla^2 j_0(\mathbf{r}'') + \frac{1}{3}\left(\frac{\hbar}{mc}\right)^2\nabla^4 j_0(\mathbf{r}'')\right]d\mathbf{r}''$$

$$= \frac{3\alpha}{8\pi^2}\int\left[U^0(r) + \frac{1}{3}\left(\frac{\hbar}{mc}\right)^2\nabla^2 U^0(r)\right]\nabla^2 j_0(\mathbf{r}'')d\mathbf{r}''$$

$$= \frac{3\alpha}{8\pi^2}\int U^1(r)\ \nabla^2 j_0(\mathbf{r}'')d\mathbf{r}'' \qquad \text{(43b)}$$

for the vector meson theory, where

$$U^1(r) = U^0(r) + \frac{1}{3}\left(\frac{\hbar}{mc}\right)^2\nabla^2 U^0(r) = \frac{1}{2\pi^2\hbar}\int\frac{f^1(k^2)}{k^2}.\exp.-\frac{i}{\hbar}(\mathbf{kr})d\mathbf{k}. \qquad \text{(44b)}$$

If we now substitute the integral (36a) for $f^0(k^2)$ in (44a) we can integrate it following Pauli and Rose and have

$$U^0(r) = -\frac{1}{3r}\left[\left(-\frac{2}{3}\xi^2+1\right)K_0(2\xi)+\frac{4\xi}{3}(\xi^2-2)K_1(2\xi)+\xi\left(-\frac{4}{3}\xi^2+3\right)B(2\xi)\right] \quad .. \; (45a)$$

where $\xi = \frac{mc}{\hbar} r = \kappa r$, $K_0(x)$ and $K_1(x)$ are the well-known integrals for the Bessel functions of the second kind *,

$$K_0(x) = \int_1^\infty \frac{e^{-qx}}{(q^2-1)^{\frac{1}{2}}}\,dq, \quad K_1(x) = x\int_1^\infty e^{-qx}(q^2-1)^{\frac{1}{2}}\,dq$$

and $B(x)$ is an indefinite integral of $K_0(x)$

$$B(x) = \int_x^\infty K_0(y)dy = \int_1^\infty \frac{e^{-qx}}{q(q^2-1)^{\frac{1}{2}}}\,dq.$$

Then from (44b) and (45a), $U^1(r)$ is obtained as

$$U^1(r) = -\frac{1}{3r}\left[-\frac{1}{3}(2\xi^2+1)K_0(2\xi)+\frac{2}{3\xi}(2\xi^4+1)K_1(2\xi)-\frac{\xi}{3}(4\xi^2-1)B(2\xi)\right]. \quad .. \; (45b)$$

The function $U^0(r)$ has a singularity at $r = 0$ and decreases very rapidly to zero for values of r different from zero. The behaviour of the function $U^1(r)$ is far worse at $r = 0$.

It can be shown that for small values of r, that is, for $\xi \ll 1$ or $r \ll \frac{\hbar}{mc}$

$$U^0(r) \sim \frac{1}{3r}\left[\gamma+\log(\kappa r)+\frac{4}{3}+O(r)\right], \quad .. \quad .. \quad .. \quad .. \; (46a)$$

and

$$U^1(r) \sim -\frac{1}{3r}\left[\frac{1}{3\kappa^2 r^2}+\log(\kappa r)+\gamma-\frac{1}{3}+O(r)\right], \quad .. \quad .. \; (46b)$$

where $\gamma = 0{\cdot}5772$ is the Euler's constant. We can also show that for large r $\left(r \gg \frac{\hbar}{mc}\right)$

$$U^0(r) \sim -\frac{\sqrt{\pi}}{8}\cdot\frac{1}{\kappa^{\frac{5}{2}}r^{\frac{7}{2}}}e^{-2\kappa r}\left[1+O(r^{-1})\right], \quad .. \quad .. \quad .. \; (47a)$$

and

$$U^1(r) \sim -\frac{7\sqrt{\pi}}{24}\cdot\frac{1}{\kappa^{\frac{5}{2}}r^{\frac{7}{2}}}e^{-2\kappa r}\left[1+O(r^{-1})\right] \quad .. \quad .. \quad .. \; (47b)$$

It should be noticed that both $U^0(r)$ and $U^1(r)$ tend to $-\infty$ as $r \to 0$, and they approach the value zero from the negative side as $r \to \infty$. $U^0(r)$ and $U^1(r)$ are negative for all values of r, that is, whatever may be the values of r' and r''. Hence the sign of the induced charge is the same as the inducing one in both the theories in those regions of space where the sign of the Laplacian of the inducing charge is negative. Further if the Laplacian of the inducing charge exists in a certain region of space, the induced charge will exist in those regions and also in a small range of space of the order of the critical length κ^{-1} surrounding those regions. To get an approximate expression for the induced charge we expand $\nabla^2 j_0(\mathbf{r}'')$ about the point $\mathbf{r}'$ by Taylor's expansion and if the field is a slowly varying one we can retain only the first term in this expansion;

* Watson, Theory of Bessel Functions, §6.3.

then since the integral of $U^0(r)$ or $U^1(r)$ over all space is $-\frac{\pi}{15}\left(\frac{\hbar}{mc}\right)^2$ we have from (43a) and (43b) as a first approximation

$$\delta j_0^0(\mathbf{r}') = -\frac{\alpha}{2\pi}\cdot\frac{1}{60}\left(\frac{\hbar}{mc}\right)^2 \nabla^2 j_0(\mathbf{r}'),$$

and

$$\delta j_0^1(\mathbf{r}') = -\frac{3\alpha}{2\pi}\cdot\frac{1}{60}\left(\frac{\hbar}{mc}\right)^2 \nabla^2 j_0(\mathbf{r}')$$

which are in agreement with the first term of (38a) and (38b) respectively.

It is already seen that there exists an induced charge distribution in the immediate neighbourhood of space varying external charges. An immediate consequence of this corresponds to a new term in the mutual potential of the external charges when the charges are very close to one another within a distance of the order of the critical length. These forces will not in general be Coulomb forces and the deviations may be computed directly from (43a) and (43b). For point charges these deviations can be easily calculated from the interaction energy matrix, and this will be done in the next article.

4. Deviation from the Coulomb Potential.

The interaction term associated with the scalar potential $A_0(\mathbf{r}')$ is given by

$$H' = e\int A_0\psi^*\beta_4\psi d\mathbf{r}' = e\sum_p\sum_{p'} \{a^*(\mathbf{p})u^+(\mathbf{p})^* + b(\mathbf{p})u^-(\mathbf{p})^*\}\beta_4$$
$$\times\{a(\mathbf{p}')u^+(\mathbf{p}') + b^*(\mathbf{p}')u^-(\mathbf{p}')\}\int A_0(\mathbf{r}')\exp.\frac{i}{\hbar}(\mathbf{p}'-\mathbf{p},\mathbf{r}')d\mathbf{r}'. \quad \ldots \quad (48)$$

Suppose there are two fixed point charges $Z'e$ and $Z''e$ at $\mathbf{r}_1$ and $\mathbf{r}_2$ separated by a distance $r = |\mathbf{r}_2 - \mathbf{r}_1|$. Under the influence of the mutual field of these charges pairs of mesons are created and annihilated. The total energy of the pair distribution can be obtained by using the perturbation matrix H'_{nk} for creation or annihilation of pairs as given by (48) from the expression

$$E' = \sum_n\sum_k \frac{|H'_{nk}|^2}{E_k - E_n}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (49)$$

One then obtains by the usual method

$$E' = -e^2\sum_p\sum_{p'}\frac{\{u^+(\mathbf{p})^*\beta_4 u^-(\mathbf{p}')\}\{u^-(\mathbf{p}')^*\beta_4 u^+(\mathbf{p})\}}{E(\mathbf{p})+E(\mathbf{p}')}\cdot\int A_0(\mathbf{r}')\exp.\frac{i}{\hbar}(\mathbf{p}'-\mathbf{p},\mathbf{r}')d\mathbf{r}'$$
$$\times\int A_0(\mathbf{r}'')\exp.\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}',\mathbf{r}'')d\mathbf{r}''.$$

On summing over the directions of polarisation in the states $\mathbf{p}$ and $\mathbf{p}'$, replacing the summations by integrations and defining $\mathbf{p}' = \mathbf{q}+\mathbf{k}/2$ and $\mathbf{p} = \mathbf{q}-\mathbf{k}/2$, we get

$$E^0 = -\frac{e^2}{4c}\cdot\frac{1}{(2\pi\hbar)^6}\int A_0(\mathbf{r}')d\mathbf{r}'\int A_0(\mathbf{r}'')d\mathbf{r}''\int d\mathbf{k}\exp.-\frac{i}{\hbar}(\mathbf{k},\mathbf{r}''-\mathbf{r}')\cdot\int\frac{(\epsilon-\epsilon')^2}{\epsilon\epsilon'(\epsilon+\epsilon')}d\mathbf{q} \quad \ldots \quad (50a)$$

for the scalar meson theory; and

$$E^1 = -\frac{3e^2}{4c}\cdot\frac{1}{(2\pi\hbar)^6}\int A_0(\mathbf{r}')d\mathbf{r}'\int A_0(\mathbf{r}'')d\mathbf{r}''\int d\mathbf{k}\exp.-\frac{i}{\hbar}(\mathbf{k},\mathbf{r}''-\mathbf{r}')\cdot\int\frac{1}{\epsilon\epsilon'(\epsilon+\epsilon')}$$
$$\times\left\{(\epsilon-\epsilon')^2 + \frac{2}{3m^2c^2}\left(q^2k^2 - (\mathbf{q}\mathbf{k})^2\right)\right\}d\mathbf{q} \quad \ldots \quad \ldots \quad \ldots \quad (50b)$$

for the vector meson theory, where ϵ and ϵ' are defined by (22). After subtraction of the singularities as before, (50a) and (50b) assume the forms:

$$E^0 = -\frac{\alpha}{128\hbar^5\pi^5}\int A_0(\mathbf{r}')d\mathbf{r}'\int A_0(\mathbf{r}'')d\mathbf{r}''\int k^2 d\mathbf{k} f^0(k^2).\exp.-\frac{i}{\hbar}(\mathbf{k},\mathbf{r}''-\mathbf{r}'), \quad .. \quad (51a)$$

and

$$E^1 = -\frac{3\alpha}{128\hbar^5\pi^5}\int A_0(\mathbf{r}')d\mathbf{r}'\int A_0(\mathbf{r}'')d\mathbf{r}''\int k^2 d\mathbf{k} f^1(k^2).\exp.-\frac{i}{\hbar}(\mathbf{k},\mathbf{r}''-\mathbf{r}'), \quad .. \quad (51b)$$

where $f^0(k^2)$ and $f^1(k^2)$ are given by (36a) and (36b) respectively. If we now introduce the potential we have from (40)

$$\int A_0(\mathbf{r}').\exp.\frac{i}{\hbar}(\mathbf{kr}')d\mathbf{r}' = \frac{4\pi\hbar^2 e}{k^2}\left[Z'\exp.\frac{i}{\hbar}(\mathbf{kr}_1) + Z''\exp.\frac{i}{\hbar}(\mathbf{kr}_2)\right],$$

$$\int A_0(\mathbf{r}'').\exp.-\frac{i}{\hbar}(\mathbf{kr}'')d\mathbf{r}'' = \frac{4\pi\hbar^2 e}{k^2}\left[Z'\exp.-\frac{i}{\hbar}(\mathbf{kr}_1) + Z''\exp.-\frac{i}{\hbar}(\mathbf{kr}_2)\right].$$

On substitution of these quantities in (51a) and (51b), the terms depending on Z'^2 and Z''^2 alone represent the contribution to the proper energy of the charges, and the product term represents the interaction energy. We then get

$$E^0_{\text{int}} = -\frac{\alpha Z'Z''e^2}{4\hbar\pi^3}\int\frac{f^0(k^2)}{k^2}\exp.-\frac{i}{\hbar}(\mathbf{kr})d\mathbf{k} = -\frac{\alpha Z'Z''e^2}{2\pi}U^0(r), \quad .. \quad (52a)$$

and

$$E^1_{\text{int}} = -\frac{3\alpha Z'Z''e^2}{4\hbar\pi^3}\int\frac{f^1(k^2)}{k^2}\exp.-\frac{i}{\hbar}(\mathbf{kr})d\mathbf{k} = -\frac{3\alpha Z'Z''e^2}{2\pi}U^1(r), \quad .. \quad (52b)$$

where $U^0(r)$ and $U^1(r)$ are defined by (44a) and (44b). Thus the mutual potential energy of two fixed point charges $Z'e$ and $Z''e$ separated by a distance r is

$$V^0(r) = Z'Z''e^2\left[\frac{1}{r} - \frac{\alpha}{2\pi}U^0(r)\right] \quad .. \quad .. \quad .. \quad (53a)$$

for the scalar meson theory, and

$$V^1(r) = Z'Z''e^2\left[\frac{1}{r} - \frac{3\alpha}{2\pi}U^1(r)\right] \quad .. \quad .. \quad .. \quad .. \quad (53b)$$

for the vector meson theory. Since $U^0(r)$ and $U^1(r)$ are negative for all values of r, the deviation from Coulomb law is always positive for like charges in both the theories.

5. Elastic Scattering of Heavy Particles.

When the distance of separation is of the order of the critical length κ^{-1}, the deviations from Coulomb field discussed in the previous article become effective and may perhaps be detected by scattering experiments. It has already been pointed out that all the calculations of the present theory are based on the assumption that the field strength should always be small compared to the critical field. If the charge on the particles is not greater than Ze then the condition $|\mathbf{E}| < \frac{m^2c^3}{\hbar e}$ means only that the distance of separation of particles must be greater than $(\alpha Z)^{\frac{1}{2}}\kappa^{-1}$. Hence in order that our calculations may be within the range of the validity of the theory the energy of the incident particle should be assumed to be such that the distance of closest approach is greater than this distance. On this supposition the deviations from the scattering for a Coulomb field are to be found in the present section.

Since the deviations from ordinary Coulomb scattering will be important only for high energetic incident particles the Born's approximation may be used. According to Born's theory the scattered amplitude for unit incident flux scattered by a fixed scattering centre is

$$\chi(\theta) = -\frac{M}{2\pi\hbar^2}\int V(r) \exp.\frac{ip}{\hbar}(\mathbf{e}-\mathbf{n}, \mathbf{r})d\mathbf{r}, \quad \ldots \quad \ldots \quad \ldots \quad (54)$$

where $\mathbf{e}$ is the unit vector in the direction of incident particle, $\mathbf{n}$ is the unit vector in the direction in which scattering is observed, θ is the angle between $\mathbf{e}$ and $\mathbf{n}$, $\mathbf{p}$ is the initial momentum and M is the mass of the incident particle. If $Z'e$ and $Z''e$ be the charges of the incident and scattering particles, then the mutual potential energy $V(r)$ is given by (53a) in the scalar meson theory and by (53b) in the vector meson theory, $V(r)$ being the Coulomb potential plus the deviation from it. On evaluation of the integral (54) we get

$$\chi^0(\theta) = -\frac{MZ'Z''e^2}{2p^2}\operatorname{cosec}^2\frac{\theta}{2}\left[1+\frac{\alpha}{6\pi}G^0(l)\right] \quad \ldots \quad \ldots \quad \ldots \quad (55a)$$

for the scalar meson theory, and

$$\chi^1(\theta) = -\frac{MZ'Z''e^2}{2p^2}\operatorname{cosec}^2\frac{\theta}{2}\left[1+\frac{\alpha}{2\pi}G^1(l)\right] \quad \ldots \quad \ldots \quad \ldots \quad (55b)$$

for the vector meson theory, where

$$G^0(l) = -3f^0(l) = l^2\int_0^1 \frac{z^4dz}{4+l^2(1-z^2)} = -\frac{4}{3}-\frac{4}{l^2}+\left(1+\frac{4}{l^2}\right)^{\frac{3}{2}}\log\left\{\left(1+\frac{l^2}{4}\right)^{\frac{1}{2}}+\frac{l}{2}\right\}, \quad \ldots \quad (56a)$$

$$G^1(l) = -3f^1(l) = \left(1-\frac{l^2}{3}\right)G^0(l), \quad \ldots \quad \ldots \quad \ldots \quad (56b)$$

$$l = \frac{2p}{mc}\sin\frac{\theta}{2}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (57)$$

where m is the mass of the meson. The first term in (55a) or (55b) gives the ordinary Coulomb scattering and the second term is the contribution to the deviation from this scattering law. For small values of l

$$G^0(l) \sim G^1(l) \sim \frac{l^2}{20}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (58)$$

and for large l

$$G^0(l) \sim -\frac{4}{3}+\log l, \quad \text{and} \quad G^1(l) \sim -\frac{l^2}{3}\left(-\frac{4}{3}+\log l\right). \quad \ldots \quad \ldots \quad (59)$$

Hence for small values of l, the deviation from the Coulomb scattering is finite and small for all values of θ in both the theories. The function $G^0(l)$ has no real zeros, while the function $G^1(l)$ has only one real zero at $l = \sqrt{3}$. For particles of a given incident velocity the function $G^0(l)$ of the scalar meson theory rises rapidly with increasing θ in the neighbourhood of $\theta = 0$ and then assumes practically constant value increasing with θ as logarithm of sin $\theta/2$ throughout the major portion of the angular range. On the other hand for particles of given incident velocity the function $G^1(l)$ of the vector meson theory rises rapidly with increasing θ in the neighbourhood of $\theta = 0$ and remains positive up to the value of θ for which $l = \sqrt{3}$ at which it attains the zero value; and afterwards it becomes negative, the absolute value increasing rapidly with θ practically as quadratic function of sin $\theta/2$ for the rest of the portion of angular range. Thus according to the scalar meson theory the deviation from ordinary scattering law is practically constant and has a small magnitude, and so it has very little chance of experimental detection; but this effect as obtained from the vector meson theory may perhaps be detected by experiment. It must again be pointed out that for very close distance of approach which is possible when the energy of the incident particle is very high, where these deviations may be large, the theory may not be applied.

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