ON THE THEORY OF RADIATION LOSS BY MESONS.

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ABSTRACT.

The radiation emitted by a fast meson moving in the field of a nucleus is investigated by the method of virtual quanta. It is found that for meson of spin one the radiation emitted is mainly due to the Compton scattering of the virtual quanta whose frequencies lie as a result of radiation reaction, in the neighbourhood of $\nu \sim \left(\frac{288}{5}\right)^{\frac{1}{2}} 137 \frac{\mu c^2}{h}$. For meson of spin zero the corresponding frequency is $\nu \sim \frac{\mu c^2}{h}$ and the effect of radiation reaction is negligible.

The radiation emitted by a meson of very high energy in the field of a nucleus has been studied by Booth and Wilson (1940), Christy and Kusaka (1941), and Kobayasi and Utiyama (1940). Since the direct calculations from the perturbation theory is very complicated, these authors have calculated the amount of energy loss by the method of virtual quanta as developed by Fermi (1924), Williams (1935) and Weiszäcker (1934). They obtained the striking fact that the energy loss of the meson due to emission of radiation varies directly as the square of its energy. Such a phenomenon, which is characteristic to the meson theory, has led to the belief that the theory in this high energy region is inapplicable in its present form. The recognition by Bhabha (1940) of the importance of the radiation damping, particularly for meson of high energy and its quantum mechanical calculations by Heitler (1941) and Wilson (1941) have, however, aroused hopes for a solution of this fundamental difficulty. The bremsstrahlung formula for meson of spin unity with the radiation damping has been obtained by Wilson (1941) but as this formula, which is very important in connection with the production of large bursts by mesons, contains some errors we propose in the following to derive approximate expressions for the bremsstrahlung cross-section from elementary considerations which, we hope, will clear the fundamental points involved in the detailed calculations. The results obtained are then compared with the more exact expressions for the cross-section which had been calculated by us (Chakrabarty and Majumdar, 1944) previously.

We apply the method of the virtual quanta to obtain approximately the radiation emitted by a fast meson in the neighbourhood of a nucleus. In the method of virtual quanta we first calculate the Compton scattering of the virtual radiation representing the field of the moving nucleus and these scattered quanta when transformed to a system in which the meson is moving past a stationary nucleus gives then the bremsstrahlung in question. Let the meson move with velocity v and energy W past a nucleus of charge Ze, the nucleus being so heavy that it can be considered to be fixed (rest system). Let us consider the process in a Lorentz frame of reference where the meson is initially at rest and the nucleus is moving past it with velocity v (moving system). Now for $v \sim c$ the Coulomb field of the nucleus is greatly contracted longitudinally so that, at not too small distances, the field is largely transverse and can by Fourier analysis be represented by light quanta travelling parallel to the direction of motion of the nucleus. The number of virtual quanta per unit area of energy ck per unit energy at a distance r from the path is approximately given by

$$\left.\begin{aligned} N(k) &= \frac{\alpha Z^2}{\pi^2 r^2 k}, \quad \text{for} \quad \frac{k}{\mu c} < \frac{\lambda}{r}\frac{W}{\mu c^2} \\ &= 0 \quad , \quad \text{for} \quad \frac{k}{\mu c} > \frac{\lambda}{r}\frac{W}{\mu c^2} \end{aligned}\right\} \qquad (1)$$

where $\lambda = \frac{\hbar}{\mu c}$, $\alpha = \frac{e^2}{\hbar c}$, $ck = h\nu$, μ = mass of the meson, k = momentum of the photon.

The total number of virtual quanta lying within k and $k+dk$ is obtained by integrating over the impact parameter r and is given by,

$$n(k)dk = \frac{2}{\pi}\alpha Z^2 \frac{dk}{k}\int_{d\lambda}^{\frac{\lambda W}{ck}} \frac{dr}{r} = \frac{2}{\pi}\alpha Z^2 \log\left(\frac{W}{dck}\right)\frac{dk}{k}. \quad \ldots \quad \ldots \quad (2)$$

Since the close impacts are mostly responsible for the total energy loss one should be careful in the choice of the lower limit of the impact parameter. If we take it to be equal to the nuclear radius then $d = \frac{5Z^{\frac{1}{3}}}{6}$. Wilson takes $d=1$. There are various approximations involved in replacing the moving Coulomb field by light quanta and cutting off the energy spectrum abruptly. The errors which are introduced thereby are at most of the order of $\frac{\mu c^2}{W}$.

We now consider the scattering of these virtual quanta by the meson and pick out those quanta which, after being scattered, have a given energy $\epsilon W (0 < \epsilon \leqslant 1)$ in the rest system in which the nucleus is at rest. In the moving system let $\mathbf{k}$ and $\mathbf{k}'$ be the momenta of the incident and scattered quanta and θ be the angle between them. The energy of the scattered quanta in the rest system is

$$ck_0 = \epsilon W = ck\left(1-\frac{v}{c}\cos\theta\right)\frac{W}{\mu c^2} \sim ck(1-\cos\theta)\frac{W}{\mu c^2} \qquad \text{for } v \sim c. \quad \ldots \quad (3)$$

The conservation of energy and momentum gives k' as

$$k' = \frac{k}{1+\frac{k}{\mu c}(1-\cos\theta)}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

We shall consider the scattering of the virtual quanta in two parts: (i) $ck < \mu c^2$, i.e. $\nu < \frac{\mu c^2}{h}$ and (ii) $ck > \mu c^2$, i.e. $\nu > \frac{\mu c^2}{h}$.

(i) When $ck < \mu c^2$ we can conveniently use the Thomson's formula for the scattering of quanta by the meson and neglect the Compton change of wavelength whence $k = k'$. The cross-section for this is given by

$$\phi = \frac{\frac{8\pi}{3}\left(\frac{e^2}{\mu c^2}\right)^2}{1+\gamma^2} \qquad \text{where } \gamma = \frac{2}{3}\alpha\frac{k}{\mu c}. \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

The factor γ^2 in the denominator is due to the effect of radiation reaction and may be neglected. The number of virtual quanta emitted in the moving system with momenta lying between k' and $k'+dk'$ is

$$n(k')dk' = \frac{8\pi}{3}\left(\frac{e^2}{\mu c^2}\right)^2 \frac{2}{\pi}\alpha Z^2 \log\left(\frac{W}{dck'}\right)\frac{dk'}{k'} \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

Since the Thomson scattering is symmetrically distributed about $\theta = \frac{\pi}{2}$ we obtain the number of scattered quanta in the rest system with momenta between k_0 and k_0+dk_0

$$n(k_0)dk_0 = \frac{16}{3}\left(\frac{e^2}{\mu c^2}\right)^2 \alpha Z^2 \log\left(\frac{W^2}{\mu c^2 dck_0}\right)\frac{dk_0}{k_0}. \quad \ldots \quad \ldots \quad (7)$$

The total energy loss is obviously

$$R_1 = \frac{16}{3}\left(\frac{e^2}{\mu c^2}\right)^2 \alpha Z^2 \mu c^2 \int_0^{\frac{W}{dc}} \log\left(\frac{W^2}{\mu c^2 dck_0}\right)\frac{dk_0}{\mu c}$$

$$= \frac{16}{3}\alpha Z^2\left(\frac{e^2}{\mu c^2}\right)^2 W \log\left(\frac{W}{d\mu c^2}\right). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

It is to be noted that the result applies for $ck \ll \mu c^2$, i.e. $ck_0 \ll W$ or $\epsilon \ll 1$. The radiative energy loss which is appreciably less than the incident energy is thus due to the Thomson scattering in the moving system. The cross-section is

$$\phi_1 = \frac{16}{3}\alpha Z^2\left(\frac{e^2}{\mu c^2}\right)^2 \log\left(\frac{W}{d\mu c^2}\right). \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

(ii) To find out the energy loss for $ck \gg \mu c^2$, i.e. $\nu \gg \frac{\mu c^2}{h}$ the Thomson scattering should be replaced by the scattering which is true for very high energy of the photon. When radiation reaction is taken into account the scattering is given by

$$\phi = \frac{\frac{5\pi}{36}\left(\frac{e^2}{\mu c^2}\right)^2\frac{k}{\mu c}}{1+\kappa^2}; \qquad \kappa = \sqrt{\frac{5}{288}}\,\alpha\,.\,\frac{k}{\mu c}. \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

Thus

$$\phi = \frac{5\pi}{36}\left(\frac{e^2}{\mu c^2}\right)^2\frac{k}{\mu c} \quad \text{for } \kappa < 1, \text{ i.e. } \frac{k}{\mu c} < A \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

$$= \frac{5\pi}{36}\left(\frac{e^2}{\mu c^2}\right)^2 A^2\frac{\mu c}{k} \quad \text{for } \kappa > 1, \text{ i.e. } \frac{k}{\mu c} > A \quad \ldots \quad \ldots \quad (12)$$

where

$$A = \sqrt{\frac{288}{5}}\,.\,137 \approx 1{\cdot}04\times 10^3. \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

It is interesting to note that the formula (12) corresponds closely to the Klein-Nishina formula for the scattering of the photon by the electron at rest, the cross-section decreasing as the incident energy of the photon is increased due to the radiation reaction. Now, when the energy of the incident photon is very high the scattering takes place almost in a direction θ given by $\theta \approx \left(\frac{\mu c^2}{ck}\right)^{\frac{1}{2}}$ and we have

$$k' \approx \frac{k}{1+\frac{k}{\mu c}\frac{\theta^2}{2}} \approx bk \qquad b \sim 1. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

The scattered energy in the rest system is accordingly

$$ck_0 = bck\frac{W}{\mu c^2}(1-\cos\theta) \approx \frac{1}{2}bW. \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

We, therefore, see that for every quantum of the radiation with energy $ck > \mu c^2$, which is scattered in the moving system, a quantum of energy $ck_0 \sim W$ is emitted in the rest system. The total energy loss is thus given by for $\frac{k}{\mu c} < A$

$$R_2 = \frac{5}{18}\alpha Z^2\left(\frac{e^2}{\mu c^2}\right)^2 W\int_{\mu c}^{A\mu c}\log\left(\frac{W}{dck}\right)\frac{dk}{\mu c}$$

$$= \frac{5}{18}\alpha Z^2\left(\frac{e^2}{\mu c^2}\right)^2 W\left[A\log\left(\frac{W}{dA\mu c^2}\right)+A\right] \quad \ldots \quad \ldots \quad (16)$$

and

$$\phi_2 = \frac{5}{18}\alpha Z^2\left(\frac{e^2}{\mu c^2}\right)^2\left[A\log\left(\frac{W}{dA\mu c^2}\right)+A\right] \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

and for $\frac{k}{\mu c} > A$

$$R_3 = \frac{5}{18}\alpha Z^2 \left(\frac{e^2}{\mu c^2}\right)^2 W A^2 \int_{A\mu c}^{\frac{W}{dc}} \log\left(\frac{W}{dck}\right) \frac{\mu c}{k}\frac{dk}{k}$$

$$= \frac{5}{18}\alpha Z^2 \left(\frac{e^2}{\mu c^2}\right)^2 W \left[A \log\left(\frac{W}{dA\mu c^2}\right) - A\right] \quad \ldots \quad \ldots \quad (18)$$

$$\phi_3 = \frac{5}{18}\alpha Z^2 \left(\frac{e^2}{\mu c^2}\right)^2 \left[A \log\left(\frac{W}{dA\mu c^2}\right) - A\right]. \quad \ldots \quad \ldots \quad (19)$$

By comparing (16) and (18) we find that the radiation emitted by the meson is mainly due to the Compton scattering of the virtual quanta whose frequencies are in the neighbourhood of $\nu \sim A\frac{\mu c^2}{h}\left(\text{i.e.} \frac{k}{\mu c} \sim A\right)$ in the moving system, the frequencies appreciably greater than this contributing very little to the energy loss due to the rapid convergence of the scattering cross-section for the increasing energy of the incident photon. The reaction of radiation is, therefore, practically equivalent to cutting off the frequency of the virtual quanta at $\nu \sim A\frac{\mu c^2}{h}$ in contrast to the electron case where the frequency $\nu \sim \frac{mc^2}{h}$ is responsible for most of the radiation loss and we are thus treading the danger zone where the validity of the theory has already become questionable. The formula (17) when compared with the corresponding expression of Christy and Kusaka which has been obtained by them by arbitrarily cutting off the frequency of the virtual quanta at $\nu \sim 137\frac{\mu c^2}{h}$ gives a much higher energy loss. There is, however, some uncertainty in the value of A due to very rough approximations involved in Wilson's calculations, and consequently the expressions for the cross-section will be liable to some errors. The above expressions are obviously valid when the energy of the incident meson is $\frac{W}{\mu c^2} > dA$. But when $\frac{W}{\mu c^2} < dA$ we can neglect the effect of the radiation reaction and the total energy loss is given by

$$R_4 = \frac{5}{18}\alpha Z^2 \left(\frac{e^2}{\mu c^2}\right)^2 W \int_{\mu c}^{\frac{W}{dc}} \log\left(\frac{W}{dkc}\right) \frac{dk}{\mu c}$$

$$= \frac{5}{18}\alpha Z^2 \left(\frac{e^2}{\mu c^2}\right)^2 \frac{W^2}{d\mu c^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

$$\phi_4 = \frac{5}{18}\alpha Z^2 \left(\frac{e^2}{\mu c^2}\right)^2 \frac{W}{d\mu c^2}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

The energy loss due to bremsstrahlung for meson of spin 0 can also be similarly calculated. At low energies for $\frac{k}{\mu c} \ll 1$ the energy loss is the same as for meson of spin one. But at high energy of the photon when $\frac{k}{\mu c} \gg 1$ the scattering is given by

$$\phi = \frac{\frac{\pi}{2}\left(\frac{e^2}{\mu c^2}\right)^2 \frac{\mu c}{k}}{1+\frac{1}{16}\alpha^2}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

The reaction of radiation is thus negligible for meson of spin 0 and cross-section has got the same forms as Klein-Nishina formula. The energy loss is given by

$$R_5 = \pi\alpha Z^2 \left(\frac{e^2}{\mu c^2}\right)^2 W \int_{\mu c}^{\frac{W}{dc}} \log\left(\frac{W}{dck}\right) \frac{\mu c}{k} \frac{dk}{k}$$

$$= \pi\alpha Z^2 \left(\frac{e^2}{\mu c^2}\right)^2 W \log\left(\frac{W}{d\mu c^2}\right) \quad .. \quad .. \quad .. \quad .. \quad (23)$$

$$\phi_5 = \pi\alpha Z^2 \left(\frac{e^2}{\mu c^2}\right)^2 \log\left(\frac{W}{d\mu c^2}\right). \quad .. \quad .. \quad .. \quad .. \quad (24)$$

The energy loss is, therefore, due to the Compton scattering of the virtual quanta whose frequencies are in the neighbourhood of $\nu \sim \frac{\mu c^2}{h}$, the contribution from the higher frequencies being negligibly small as in the case of electron.

It has been recently shown (Chakrabarty and Majumdar, 1944) that by introducing properly the effect of radiation damping on the emission of photons by meson the differential cross-section $\phi(\epsilon)d\epsilon$ for the emission of a photon of energy lying between ϵW and $(\epsilon+d\epsilon)W$ by a meson of spin 1 and energy W is given by the following expressions:

$$\phi(\epsilon)d\epsilon = \alpha Z^2 \left(\frac{e^2}{\mu c^2}\right)^2 d\epsilon \left[C_0 f_0(a) - C_1 f_1(a) + L(W, b)\right] \quad \text{for } a \leqslant 1 \quad .. \quad (25)$$

$$\phi(\epsilon)d\epsilon = \alpha Z^2 \left(\frac{e^2}{\mu c^2}\right)^2 d\epsilon \left[C_0 \left\{\frac{\pi}{2}\log a + f_0\left(\frac{1}{a}\right)\right\} - C_1 \left\{\frac{\pi^2}{6} + 2\log^2 a - f_1\left(\frac{1}{a}\right)\right\} + L(W, b)\right]$$

$$\text{for } a \geqslant 1 \geqslant \frac{W\alpha Z^{\frac{1}{3}} m}{A\mu c^2 \mu} \quad .. \quad (26)$$

where

$$L(W, b) = 2C_1 \log^2\left(\frac{a}{b}\right) - C_0 f_0(b) + C_1 f_1(b) + \left\{\frac{3C_2+8}{3bA} + 2C_1 \log(1+b^2) - C_0 \tan^{-1} b\right\}$$

$$\times \log\left(\frac{a}{b}\right) - \frac{(9C_2+26)}{9AB} \quad .. \quad (27)$$

$$f_0(a) = a\left(1 - \frac{a^2}{3^2} + \frac{a^4}{5^2} - \dots\right); \quad f_1(a) = a^2\left(1 - \frac{a^2}{2^2} + \frac{a^4}{3^2} - \dots\right) \quad .. \quad .. \quad (28)$$

$$C_0 = \frac{(2-2\epsilon+7\epsilon^2)A}{12} - \frac{C_2}{A}; \quad C_1 = \frac{(34-34\epsilon+7\epsilon^2)\epsilon}{48(1-\epsilon)}; \quad C_2 = \frac{(14-14\epsilon+\epsilon^2)\epsilon^2}{24(1-\epsilon)^2} \quad .. \quad (29)$$

$$a = \frac{W}{Ad\mu c^2}, \quad b = \frac{\epsilon}{2A(1-\epsilon)}. \quad .. \quad .. \quad .. \quad (30)$$

Hence the total cross-section is

$$\phi = \frac{5}{18}\alpha Z^2 \left(\frac{e^2}{\mu c^2}\right)^2 \frac{W}{d\mu c^2} \quad \text{for } a \leqslant 1 \quad .. \quad .. \quad .. \quad .. \quad (31)$$

$$\phi = \frac{5}{18}\alpha Z^2 \left(\frac{e^2}{\mu c^2}\right)^2 A \frac{\pi}{2} \log\frac{W}{dA\mu c^2} \quad \text{for } a \geqslant 1 \geqslant \frac{W\alpha Z^{\frac{1}{3}} m}{A\mu c^2 . \mu}. \quad .. \quad (32)$$

It is now evident that a combination of (17) and (19) compares with (32) and (21) with (31). The expression given by Kusaka* (1943) is only valid for very high energies, viz.

$$W \gg \frac{10\mu c^2}{\alpha^2 Z^{\frac{1}{3}}} \frac{\mu}{m} \approx 5{\cdot}2 \times 10^{14} \text{ e.v.} \quad .. \quad .. \quad .. \quad .. \quad (33)$$

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* There is an error in the numerical factor of the expression given by Kusaka. The numerical coefficient in the expression for $\phi(\epsilon)$ will be $\frac{1}{12}$ and not $\frac{5}{144}$. The author is indebted to Dr. Kusaka for sending him a copy of his paper before publication.

whence we obtain

$$\phi(\epsilon)d\epsilon = \frac{1}{12}\alpha Z^2\left(\frac{e^2}{\mu c^2}\right)^2\frac{\pi}{2\alpha}\left(\frac{288}{5}\right)^{\frac{1}{2}}(2-2\epsilon+7\epsilon^2)\log\frac{137\mu}{Z^{\frac{1}{3}}m}\,d\epsilon \quad \ldots \quad (34)$$

and

$$\phi = \frac{5}{18}\alpha Z^2\left(\frac{e^2}{\mu c^2}\right)^2\frac{\pi}{2}A\log\frac{137\mu}{Z^{\frac{1}{3}}m}. \quad \ldots \quad (35)$$

This expression has, therefore, no significance for the calculations of the frequency of large bursts, containing even 10,000 or more particles, and may have some significance for much larger bursts.

It is to be carefully noticed that the formula (26), which is correct and agrees with the corresponding formula derived in the previous section from elementary considerations, is in fact dependent entirely on the choice of the lower limit of the impact parameter, the contribution from the upper limit being negligibly small. The energy loss is, therefore, mostly due to close collisions and it is very important to take into account the modification of the Coulomb field within the nuclear radius given by $r = d\lambda$. A lower limit of the cross-section, which is perhaps correct so far as the order of magnitude is concerned, may, however, be obtained if we assume that the Coulomb field is smoothed out in the nucleus whose radius is given by $r = \frac{5}{6}Z^{\frac{1}{3}}\lambda$, i.e. by neglecting the energy loss due to nuclear collisions, which has been shown by Weiszäcker to be very small for the case of electron. In the approximations of Wilson and Kusaka this property of the formula is lost and the contribution from the distant collisions is quite significant. This is due to the very rough approximations involved in their analysis and also for fixing up a very high lower limit for W given by (33). The question of the choice of the lower limit will affect the result more for the energy of the meson given by $\frac{W}{\mu c^2} < dA$ when radiation damping can be neglected than for higher energies where the unknown factor is involved in the logarithmic terms.

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