ON MAINTAINED VIBRATIONS.

By R. N. Ghosh, *D.Sc., F.N.I., Lecturer in Physics, Allahabad University, Allahabad.*

(*Received January 25; read April 17, 1946.*)

Abstract.

This paper discusses the maintained vibration of the systems which are self-excited. Harmonium reed has been taken as a standard case for illustration. It is found that the maintenance of vibration is due to partial vacuum in the rear of the reed, a phenomenon which is well known in the case of a disc exposed to a uniform stream of air. This is true even for small values of the pressure in the wind chest. The necessary condition is that the vibrations must grow, i.e. the damping coefficient should be negative for small displacements.

In many acoustical cases the maintenance of vibrations of a system can be traced as due to the effect of a periodic force acting upon the system. There are many cases, however, where the vibration of the system is self-maintained; that is, the vibration of the system causes such changes in the associated parts of the system that by their mutual effect the vibration is maintained. In such cases the forces that arise can be written down as

$$f = \psi(x, v)v \quad \ldots \quad (1)$$

where x and v represent the displacement and velocity of the system. The equation of motion of the system then takes the form

$$m\ddot{x}+r\dot{x}+sx = \psi(x, v)v. \quad \ldots \quad (2)$$

We have to examine under what circumstances the vibration of the body can be maintained. We write equation (2) in the form

$$v\frac{dv}{dx}+\omega^2 x = \{\psi(x, v)-\mu\}v \quad \ldots \quad (3)$$

where $s/m = \omega^2$, $r/m = \mu$.

Integrating (3) with respect to x, we get

$$v^2+\omega^2x^2 = \oint \{\psi(x, v)-\mu\}vdx+c^2. \quad \ldots \quad (4)$$

Let us consider the case when both

$$\psi = \mu = 0$$

then equation (4) becomes $\qquad v^2+\omega^2x^2 = c^2 \quad \ldots \quad (5)$

c^2 being a constant.

If we represent v as ordinate and x as abscissa and plot equation (5), we get a series of ellipses for different values of c. It is well known that under these circumstances equation (2) predicts a vibration of the system with frequency $\omega/2\pi$.

Let us now consider that ψ and μ have finite values and enquire in what circumstances v and x will form a closed curve. Assuming this to be the case, we integrate

(3) over a complete cycle, then the left-hand members of (4) vanish because we come back to the same values of v and x, hence

$$\oint \{\psi(x, v)-\mu\} v dx = 0$$

or since $dx = vdt$, we find

$$\oint \{\psi(x, v)-\mu\} v^2 dt = 0. \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

This indicates that if v and x form a closed curve, then equation (6) must hold true. That is, if the system performs free vibrations without damping, then v, x when represented as a curve must form a closed curve, and equation (6) when integrated over the complete cycle must vanish. Since the integral (6) has to vanish, the function

$$\{\psi(x, v)-\mu\}$$

must have both positive and negative values over the cycle in order that their sum over the complete cycle might be zero. Thus from the nature of the function ψ one can predict *whether it will have both positive and negative values during the cycle and the possibility of their sum to vanish.* This is then the condition that must be satisfied in order a system might perform free vibrations without damping. A special type of ψ that arises in connection with the maintenance of electrical oscillation of a valve circuit is given by

$$\psi = \mu(1-x^2)v.$$

It is apparent that ψ will have both positive and negative values according as x is less than or greater than unity. A detailed discussion (1) with graphical demonstration will be found in the Bulletin of American Mathematical Society (Karman, 1940).

We shall consider the case of a reed vibrator of the harmonium (Ghosh, 1945). It is well known that the vibration of the reed is maintained by the periodic efflux of air from the wind chest, and the periodic puffs of air give rise to pressure waves that spread outwards as sound waves. The equation of motion of the reed (Das, 1932) is obtained by taking moment about the fixed edge of the reed

$$ml\ddot{x}+rl\dot{x}+lsx = F(p)\,\frac{l}{2} \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

where m, r and s represent their usual effective values for the reed, and l represents the length of the reed. $F(p)$ represents the force upon the reed due to difference of pressure between the front and the rear surface of the reed. x is taken positive outward. The above equation is written in the form

$$\ddot{x}+\mu\dot{x}+\omega^2 x = \frac{F(p)}{2m} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

where $\omega/2\pi$ represents the natural frequency of the reed, and μ its damping coefficient.

Determination of p:—The average pressure P in the wind chest is kept a few centimetres of water above normal atmospheric pressure. In the mean position of rest the reed is seated in the slot closing all opening round the periphery of the reed.[1] When the reed moves outward a small chink allows air to rush out to the

[1] There may be other cases when at the normal position of the reed it is slightly bent upwards, and a small chink is left over the slot allowing a little flow of air at the mean position of rest.

outside atmosphere. As the outward displacement of the reed increases, the chink width increases in proportion and allows more air to rush outwards. When the reed moves inward from its extreme outward position the chink width diminishes and finally closes at the normal position of the reed, and flow of air ceases. This state of affairs continues during the interval the reed completes half vibration in the inward direction. Let us suppose that the average pressure in the wind chest is P; when air issues outwards through the chink, the pressure in the wind chest falls by an amount p, while in the outside region where air flows, the pressure rises by the same amount. Consequently the flow of air takes place under a difference of pressure

$$(P-2p)$$

and if we assume Bernoulli's law the average flow velocity q will be given by

$$q = \sqrt{\frac{2P}{\rho}\left(1-\frac{2p}{P}\right)}$$

where ρ represents the density of air. Since p varies, the velocity of efflux q varies. The volume of air coming out of the chink at any instant is given by

$$Q = bxq, \quad x > 0$$

where b represents the effective length of the chink and x its width. It will be remembered that x is the displacement of reed outwards, and Q is zero.

$$Q = 0, \quad x \gtreqless 0.$$

If we consider the air near the reed vibrator to behave like an incompressible fluid then the flux of air at a distance R from the reed over the hemispherical surface $2\pi R^2$ will be equal to the volume of air issuing out of the chink per second at any instant, hence

$$2\pi R^2 u = qbx$$

where u represents the particle velocity in that region. If p represents the increase of pressure in the outside atmosphere due to efflux of air, then the region near the reed vibrator acts like a source of sound and we have then

$$p = f(ct-r)/r \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

representing pressure wave from the source. The equation for pressure can be written in the form

$$-\frac{dp}{dr} = \frac{\rho}{2\pi R^2}\frac{dQ}{dt} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

R representing, as before, the radius of the hemispherical surface of the source. From (9) and (10) we get

$$\frac{f'(ct-R)}{R} + \frac{f(ct-R)}{R^2} = \frac{\rho}{2\pi R^2}\frac{dQ}{dt}$$

i.e.

$$\frac{dp}{dt} + \frac{c}{R}p = \frac{Z_0}{2\pi R^2}\cdot\frac{dQ}{dt} \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

The above equation holds good only for the period when the reed moves outwards and chink is open. When the chink is closed, the pressure p becomes zero in the outside atmosphere and remains as such till the chink begins to open again. The pressure within the wind chest in the region near the reed vibrator, in the former interval falls down by an amount p, and in the latter period the pressure attains the average value P. Assuming Q to vary as $e^{j\omega t}$ we get

$$p = \frac{Z_0}{2\pi R^2} \cdot \frac{j\omega Q}{(c/R+j\omega)} \cdot \quad \ldots \quad (12)$$

Substituting the value of Q we get finally

$$p = \frac{Z_0 bk^2}{2\pi} \sqrt{\frac{2P}{\rho}\left(1-\frac{2p}{P}\right)} \cdot (x+\beta v)/(1+k^2R^2) \quad \ldots \quad (13)$$

where $\beta = c/R\omega^2$.

Writing α for

$$\alpha = \frac{Z_0 bk^2}{2\pi} \sqrt{\frac{2P}{\rho}} \quad \ldots \quad (14)$$

we find

$$p = \alpha\left(1-\frac{2p}{P}\right)^{\frac{1}{2}} \chi \quad \ldots \quad (15)$$

where $\chi = (x+\beta v)$ and solving for p we find

$$p = \alpha\chi\left(1+\frac{\alpha^2\chi^2}{P^2}\right)^{\frac{1}{2}} - \frac{\alpha^2\chi^2}{P}. \quad \ldots \quad (16)$$

Determination of $F(p)$:—The normal force upon a flat plate exposed to a stream of air has been the subject of classical investigations by Lord Rayleigh and Kirchoff. According to them, it is given by

$$\frac{2\pi}{4+\pi} \frac{V_0^2\rho}{2} \Big/ cm^2$$

where V_0 represents the velocity of the undisturbed stream and ρ its density. The above formula has been derived on the assumption that the air breaks away from the plate at the sharp edges and leaves a 'dead' air region behind the plate throughout which the pressure is uniform and equal to that in the undisturbed stream, and (2) that the pressure and velocity in the free surfaces separating the stationary from the moving fluid are equal to those in the undisturbed fluid. Recent experiments of Fage (1927) with flat plate in the wind tunnel have revealed that both these assumptions are not realised in practice. The velocity V_1 of the air separating the 'dead' region from the streaming portion is larger than the velocity of the stream over the undisturbed region, i.e. $V_1 > V_0$. The flow round the plate closes in behind it, and owing to the velocity component parallel to the plate a large amount of suction is produced with the result that the pressure in the rear is considerably less than that in the undisturbed region. Consequently the total force contributed partly by the dynamical pressure in front, and partly by the defect of pressure in the rear, is much larger than that obtained from Rayleigh's formula. Fage and Johansen's experiments in wind tunnel show that

$$\int (p-p_0)dS/(V_0^2\rho/2) = 2{\cdot}13$$

where p represents the pressure at any point of the surface of the plate, and the integral is taken both over the front and rear surface, and V_0 represents the undisturbed velocity of the stream of air; also they found that

$$p_0 + \frac{1}{2} V_0^2\rho = p_m + \frac{1}{2} V_m^2\rho$$

where p_0 and p_m represent the pressure in the undisturbed and the 'dead water' regions respectively, and V_m represents the velocity at the surface of discontinuity separating the 'dead water' region. We obtain at once

$$p_0 - p_m = \frac{1}{2} V_0^2 \rho \left(\frac{V_m^2}{V_0^2} - 1 \right).$$

$(V_m/V_0)^2$ according to their experiments is 2·38, while the dynamical pressure p_1 in front is given by

$$p_1 - p_0 = \frac{1}{2} V_0^2 \rho$$

hence

$$p_1 - p_m = \frac{1}{2} V_0^2 \rho \cdot 2{\cdot}38.$$

The effective area of the 'dead water' region behind the plate is much larger than the surface over which the dynamical pressure p_1 is effective. Thus on account of the large surface of the dead water region and the defect of pressure prevailing over that region the total force is much larger than that given by the above equation.

In the present case of the reed and the chink the air flows past the edges of the reed practically in the same manner, and consequently the region behind the reed must be one of defect of pressure. The details of the phenomenon are not known at present but we shall make use of the results obtained in the case of flat plate. We have assumed that the average flow velocity q is given by equation (1), the velocity near the chink is represented by V_0 and the pressure by p_0, while at the surface of discontinuity the velocity and pressure are given by V_m and p_m respectively; while at a small distance ahead, the pressure is p; and since a lot of mixing takes place over this region, the velocity is small in comparison to V_0, and V_0 is very much larger than q. We therefore obtain the following equations for potential flow :—

$$\frac{1}{2} V_0^2 \rho + p_0 = p$$

and

$$\frac{1}{2} V_m^2 \rho + p_m = p$$

since both streams mix over the region where the pressure is p; we obtain at once

$$p_m - p_0 = -(p - p_0) \left\{ \frac{V_m^2}{V_0^2} - 1 \right\}. \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

We assume that V_m is much greater than V_0 as found by Fage and Johansen in the case of the flat plate. We therefore write

$$p_m - p_0 = -pf \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

since p_0 is small, and f is a numerical constant greater than unity. Consequently $F(p)$ will be modified to the form

$$F(p) = S(P - p) + S'fp$$

where S represents the effective surface in front and S' the same in the rear, and $S' \gg S$; or

$$F(p) = S\{P + ff'p - p\}$$

where $S'/S = f' > 1$.

We write

$$F(p) = S\{P + f_2 p\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

where $f_2 \sim 2$. Writing the approximate expression for

$$p = \alpha(x+\beta v) - \frac{\alpha^2}{P}(x^2+2\beta xv) \quad \ldots \quad (20)$$

we get

$$F(p) = SP + Sf_2\left(\alpha x - \frac{\alpha^2 x^2}{P}\right) + Sf_2\alpha\beta\left(1-\frac{2\alpha x}{P}\right)v. \quad \ldots \quad (21)$$

The above expression for $F(p)$ holds good for positive values of x. It does not hold for negative values of x, i.e. when the flow stops. It will be observed that if we put $x = 0$, $F(p)$ reduces to

$$F(p) = SP + Sf_2\alpha\beta v$$

when the reed is vibrating, while when the reed is at rest we have

$$F(p) = SP.$$

On substituting this value in (8) we get

$$\ddot{x}+\omega^2 x = \frac{PS}{2m} + \frac{S}{2m}\left\{f_2\left(\alpha x - \frac{\alpha^2 x^2}{P}\right) + f_2\alpha\beta\left(1-\frac{2\alpha x}{P}\right)\right\} - \mu v. \quad \ldots \quad (22)$$

Collecting together the relevant terms and rewriting we obtain

$$v\frac{dv}{dx} + \omega^2 x\left(1-\frac{f_2 x}{2m\omega^2}\right) = \frac{PS}{2m} - \frac{Sf_2\alpha^2 x^2}{2mP} + \left\{\frac{Sf_2}{2m}\alpha\beta\left(1-\frac{2\alpha x}{P}\right) - \mu\right\}v.$$

On leaving out terms of small order for the present we find

$$v\frac{dv}{dx} + \omega^2 x = \frac{PS}{2m} + \left\{\frac{Sf_2}{2m}\alpha\beta\left(1-\frac{2\alpha x}{P}\right) - \mu\right\}v. \quad \ldots \quad (23)$$

We first reckon the order of magnitude of the various constants involved in the above equation.

$$Z_0 = 42;\ b = 2;\ c = 3{\cdot}4\times 10^4$$

then

$$\beta = \frac{C}{R\omega^2} = 1{\cdot}7\times 10^4/\omega^2 \text{ when } R = 2$$

and

$$\alpha = \frac{42\,.\,2\,.\,\sqrt{P}\omega^2}{2\pi\,.\,3{\cdot}6\times 10^{-2}\times c^2} = 4{\cdot}58\times 10^{-7}\sqrt{P}\omega^2. \quad \ldots \quad (24)$$

Also

$$\alpha\beta = {\cdot}77\sqrt{P}\,.\,10^{-2} \qquad \text{independent of frequency,}$$

$$\frac{Sf_2}{2m} = 6{\cdot}2.$$

We shall now consider in detail the bracketed term containing v as a factor. This whole term may be regarded as the damping coefficient. It is apparent that if we apply the criterion mentioned in page 206, the integral

$$\int \left\{\frac{S}{2m}f_2\alpha\beta\left(1-\frac{2\alpha x}{P}\right) - \mu\right\} v^2 dt \quad \ldots \quad (25)$$

over the outward half-cycle must have a positive value. Let us fix the value of ω equal to $2\pi\times 10^3$, then α/P equals

$$\frac{\alpha}{P} = 18{\cdot}32/\sqrt{P}, \quad \mu = 1.$$

The integral

$$\int \left\{\cdot 047\sqrt{P}\left(1-\frac{2x\times 18\cdot 32}{\sqrt{P}}\right)-1\right\} v^2 dt \qquad \ldots \quad \ldots \quad (26)$$

or

$$\int \{\cdot 047\sqrt{P} - 1\cdot 72x - 1\}\, v^2 dt. \qquad \ldots \quad \ldots \quad \ldots \quad (27)$$

Let $P = 2\times 10^3$, then the integral becomes

$$\int (2\cdot 2 - 1\cdot 72x - 1) v^2 dt$$

or

$$\int (1\cdot 2 - 3\cdot 4x) v^2 dt. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

It is apparent that the damping coefficient represented by the bracketed term has to be considered as negative as opposed to $-\mu$; and the damping coefficient is variable. It has a large value when x is zero, and decreases with x and tends towards the value zero. The positive value of the integral over the outward half-cycle must balance the negative value of the integral over the inward half-cycle; this latter integral reduces to

$$\int -\mu v^2 dt. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

This integral can be combined with the former one and finally we must have

$$\int (2\cdot 2 - 1\cdot 7\,.\,2x - 2) v^2 dt$$

i.e.

$$\int (\cdot 2 - 3\cdot 4x) v^2 dt \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

and this integral must vanish when summed up over the outward half-cycle. It is observed that the damping coefficient vanishes for $x = \cdot 2/3\cdot 4$, i.e. $x = \cdot 06$ cm. Up to $x = \cdot 06$ cm. it has a positive value, while beyond this value it becomes negative; in order that the integral might vanish x must therefore increase beyond x' ($\cdot 06$ cm.), the actual limiting value can be determined by solving the equation

$$v\frac{dv}{dx}+\omega^2 x = \left\{\frac{S}{2m} f_2 \alpha\beta\left(1-\frac{2\alpha x}{P}\right)-\mu\right\} v.$$

In this equation μ is not constant, the resistance is actually given by

$$\mu(v-q)$$

or

$$\mu v\left(1-\frac{q}{v}\right) = \mu' v \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

where μ' is a function of v and q, but we have for simplicity assumed

$$\mu' = \mu.$$

Growth of oscillation:—It is important to realise the significance of p_m in connection with the development of vibrations of the reed when the key is pressed and the reed is released for vibration. Consider the equation

$$\ddot{x}+\omega^2 x = -\mu\dot{x}$$

i.e.

$$v\frac{dv}{dx}+\omega^2 x = -\mu v. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

It is well known that in the case of a system represented by the above equation, if we start with a given value of v, it will die down exponentially after a few vibra-

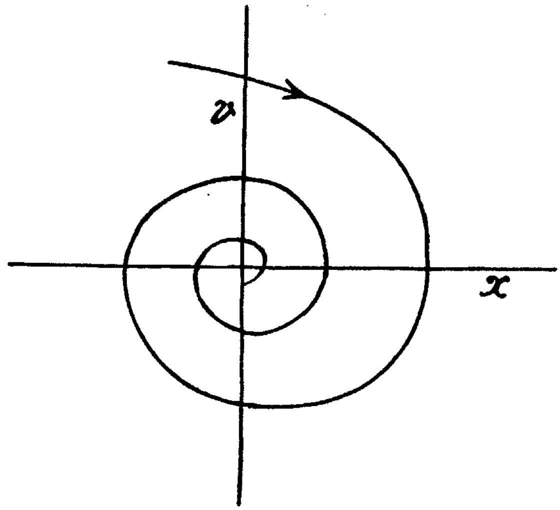

FIG. 1.

tions. If we plot v and x we shall get a logarithmic spiral closing down towards the origin (Fig. 1).

On the other hand, if the equation to the system is

$$v\frac{dv}{dx}+\omega^2 x = \mu v \quad \text{.. .. .. ..} \quad (33)$$

i.e. μ is negative, and we start with a small value v, then in the vx plane v will go on increasing according to the logarithmic spiral shown in Fig. 2.

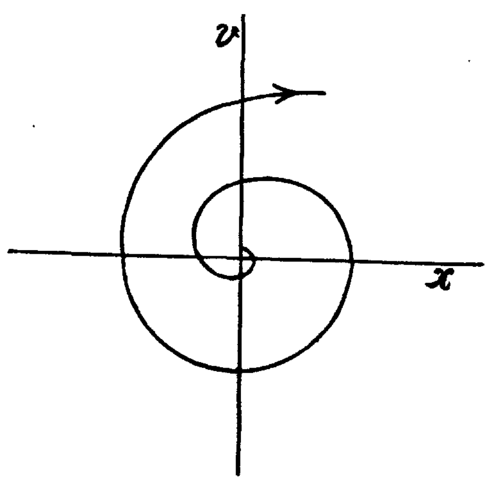

FIG. 2.

In the light of these results let us now consider the case of a reed vibrator which at the position of rest completely closes the chink.

We make use of the equation

$$v\frac{dv}{dx}+\omega^2 x = \left\{\frac{S}{2m}f_2\alpha\beta\left(1-\frac{2\alpha x}{P}\right)-\mu\right\}v \quad \ldots \quad \ldots \quad \ldots \quad (34)$$

and observe that if the conditions are favourable the first term

$$\frac{S}{2m}f_2\alpha\beta\left(1-\frac{2\alpha x}{P}\right)$$

for very small values of x, i.e. $x \sim 0$, must be greater than μ

i.e. $$\frac{S}{2m}f_2\alpha\beta > \mu.$$

Then the vibrations will grow in the manner shown by the logarithmic spiral (Fig. 2). With increasing value of x, the value of the first term diminishes till it becomes equal to

$$\frac{S}{2m}f_2\alpha\beta\left(1-\frac{2\alpha x}{P}\right) = \mu.$$

This does not represent the limiting value of x, and x increases further and the right-hand side becomes negative. The subsequent course of events is then determined by the conditions described in detail in the previous section. The v, x curve then forms a closed curve, and then the system performs free vibrations. It may be pointed out that at the initial stages when the reed is just released, we may neglect $\omega^2 x$, the equation then reduces to

$$\frac{dv}{dx} = \left\{\frac{S}{2m}f_2\alpha\beta\left(1-\frac{2\alpha x}{P}\right) - \mu\right\}$$

i.e. $$v = \left\{\left(\frac{S}{2m}f_2\alpha\beta-\mu\right)x-\frac{S}{2m}f_2\frac{\alpha^2\beta x^2}{P}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (35)$$

showing that v and x rise gradually; both v and x are zero at the beginning.

In contrast to the case discussed above let us consider that the position of rest is *slightly displaced upwards*, so that there is a small chink left over, and the flow begins with a finite value; under these circumstances the equation becomes

$$v\frac{dv}{dx}+\omega^2 x = \left\{\frac{S}{2m}f_2\alpha\beta\left(1-\frac{2\alpha(x+x_0)}{P}\right)-\mu v\right\}++ \quad \ldots \quad \ldots \quad (36)$$

where x_0 denotes the position of the reed as measured from the slot, also represents the width of the chink at the rest position of the reed. The effect of the finite chink at the rest position is to decrease the value of the negative damping coefficient, and reduce the value of x' and also the amplitude of the reed. v rises with x much more slowly in this case than in the former.

The actual nature of the maintained vibration can be obtained by solving the equation. Since the solution does not seem to be at hand, we shall draw the isoclynes

$$\frac{dv}{dx} = -\frac{\omega^2 x}{v}+\left\{\frac{S}{2m}f_2\alpha\beta\left(1-\frac{2\alpha x}{P}\right)-\mu\right\} \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

in the v, x plane; several of these isoclynes may be drawn, viz. $\frac{dv}{dx} = \cdot 5,\ 1,\ 2,$ etc., also $\frac{dv}{dx} = -\cdot 5, -1$, etc., and we can indicate, by means of short lines, the direction the integral curve must have when it crosses an isoclyne. Figure 3 shows the result

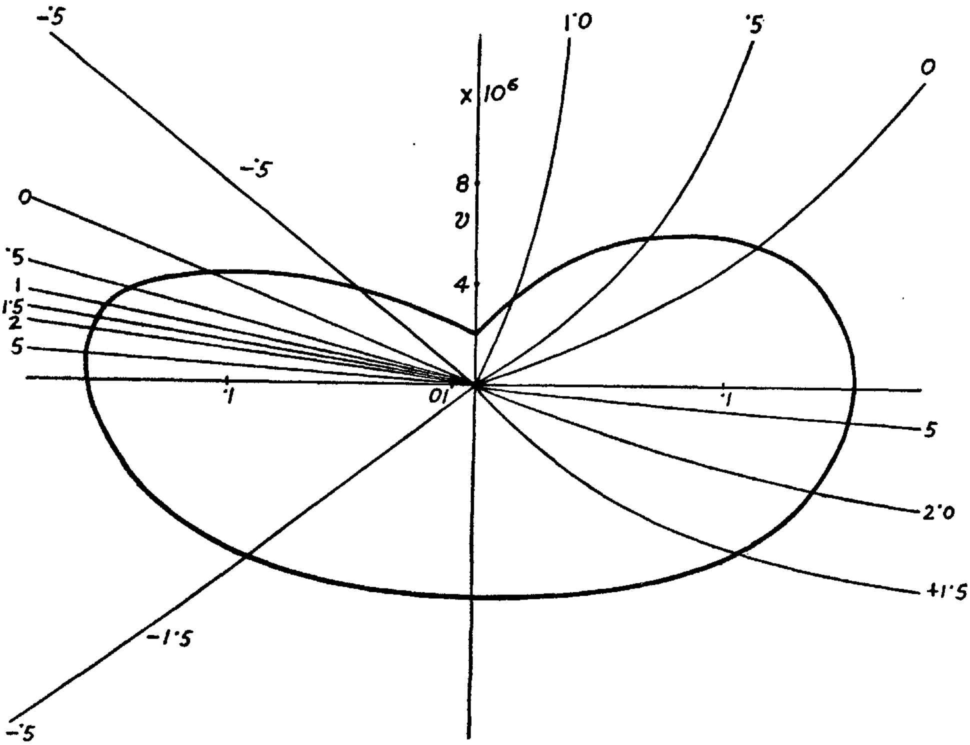

Fig. 3.

of joining together the short lines, and the full curve shows the integral curve for a given value of the pressure P, viz. $P = 2 \times 10^3$.

$$\alpha = 5{\cdot}34 \times 10^3 ; \; \frac{f_2 S \alpha^2}{2mP} = 2{\cdot}48 \times 10^3 ; \; \omega^2 = 4 \times 10^7 ; \; \mu = 1.$$

And
$$v = \frac{\omega^2 x}{\left(2{\cdot}2 - 3{\cdot}4x - 1 - \frac{dv}{dx}\right)} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (38)$$

for positive values of x. While for negative values of x

$$v = -\frac{\omega^2 x}{\left(1 + \frac{dv}{dx}\right)}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (39)$$

We take various values of $\frac{dv}{dx}$ and draw the isoclynes; then small straight lines are drawn on given points showing the direction which the integral will have at the point; then these lines are joined together (Fig. 3) to form a complete curve. Figure 3 shows the complete integral curve. It will be observed that the curve consists of two portions; for negative values of x the curve is a portion of the converging logarithmic spiral, while on the positive side v increases first for small values of x, then the curve turns round, and reaches a limiting value when v is zero. v then becomes negative and attains a maximum value when $x = 0$; beyond this x is negative and the curve follows the logarithmic spiral.

It does not seem possible to evaluate the limiting value of x otherwise. For different values of P we shall have to draw the integral curve to obtain its value.

It is, however, possible to obtain an upper limit to the maximum value of x for a given pressure P maintained within the wind chest.

The maximum value of p :—The evaluation of the maximum value of p can be made from the following considerations. Since the work done per second by the outgoing fluid must be equal to the energy which escapes through the surface of the hemisphere $2\pi R^2$ plus the spent in raising the kinetic energy of the fluid contained within the same hemisphere per second, we must have

$$(P-2p)Q = p2\pi R^2 u + \frac{d}{dt}\,\tfrac{1}{2}(\tfrac{2}{3}\pi R^3 \rho u^2)$$

i.e.

$$P-2p = p + \frac{\rho}{6\pi R}\frac{dQ}{dt}. \qquad (40)$$

Now we know that p reaches its maximum value when $\frac{dQ}{dt} = 0$.

Hence

$$P-2p_m = p_m \qquad (41)$$

i.e.

$$p_m = \frac{P}{3}.$$

Thus the limiting value of p is $P/3$; in actual practice p will be less than $P/3$ on account of eddies behind the reed. Since

$$p = \alpha\left(1-\frac{2p}{P}\right)^{\frac{1}{2}} \cdot (x+\beta v)/(1+k^2R^2)$$

and $v = 0$ when x is maximum, we have

$$x = P/\alpha\sqrt{3}, \text{ approximately.} \qquad (42)$$

References.

Von Karman, Theodore (1940). On Non-linear Problems in Engineering. *Bull. American Math. Soc.*, **46**, 619.

Ghosh, R. N. (1945). Acoustics of the Harmonium. *Proc. Nat. Inst. Sci. India*, **11**, 96.

Das, P. (1932). On Maintained Vibrations of the Harmonium. *Indian Physico-Math. Journ.*, **3**, 27.

Fage and Johansen (1927). On the flow of air behind an inclined plate. *Proc. Roy. Soc.*, **116**, 170.