A NOTE ON MULTIPLICATIVE FUNCTIONS

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§1. An Arithmetical function $f(n)$ is multiplicative if

$$f(m\,n) = f(m)\,f(n)$$

when m is prime to n. We introduce the

Definition. The function $f(n)$ is multiplicative (mod k) *if*

$$f(m\,n) \equiv f(m)\,f(n) \pmod{k}$$

whenever m is prime to n, and both are prime to k.

Let $\sigma_k(n)$ denote the sum of the kth powers of the divisors of n. Also, with Ramanujan we define $\tau(n)$ by

$$\sum_{1}^{\infty} \tau(n)x^n = x\left\{(1-x)(1-x^2)(1-x^3)\ldots\right\}^{24}$$

Bambah and I have recently proved* that

Theorem 1.—

$$\tau(n) \equiv 5n^2\sigma_7(n) - 4n\sigma_9(n) \pmod{125} \text{ if } n \text{ is prime to } 5.$$

It immediately follows that

Theorem 2.—*The function*

(1) $f(n) = 5n^2\sigma_7(n) - 4n\sigma_9(n)$ *is multiplicative* (mod 125).

The object of this note is to give a direct proof of theorem 2.

§2. From (1) we have when m is prime to n, and mn is prime to 5,

$$\begin{aligned} f(m\,n) - f(m)\,f(n) &= \{5m^2n^2\sigma_7(mn) - 4mn\sigma_9(mn)\} \\ &\quad - \{5m^2\sigma_7(m) - 4m\sigma_9(m)\}\{5n^2\sigma_7(n) - 4n\sigma_9(n)\} \\ &= -20m^2n^2\sigma_7(mn) - 20mn\sigma_9(mn) \\ &\quad + 20\{m^2n\sigma_7(m)\sigma_9(n) + mn^2\sigma_7(n)\sigma_9(m)\}. \end{aligned}$$

To prove our result, it therefore only remains to show that if x is prime to y, and xy prime to 5, we must have $xy\ \sigma_7(xy) + \sigma_9(xy) \equiv x\sigma_7(x)\sigma_9(y) + y\sigma_7(y)\sigma_9(x)$ (mod 25)
i.e.

$$x\sigma_7(x)\{y\sigma_7(y) - \sigma_9(y)\} + \sigma_9(x)\{\sigma_9(y) - y\sigma_7(y)\} \equiv 0 \pmod{25}$$

* In a joint paper communicated to the London Mathematical Society.

i.e.

$$\{x\sigma_7(x)-\sigma_9(x)\}\{y\sigma_7(y)-\sigma_9(y)\} \equiv 0 \pmod{25}. \tag{2}$$

(2) is an immediate consequence of the

Lemma

$$n\,\sigma_7(n)-\sigma_9(n) \equiv 0 \pmod 5, \text{ when } n \text{ is prime to } 5$$

proved as follows :

$$n\,\sigma_7(n)-\sigma_9(n) \equiv n \sum_{d/n} d^7 - \sum_{d/n} d^9$$

$$\equiv n \sum_{\substack{d/n \\ d \not\equiv 0(5)}} \frac{1}{d} - \sum_{\substack{d/n \\ d \not\equiv 0 \pmod 5}} d \pmod 5 \equiv 0 \pmod 5,$$

using $d^7 \equiv \frac{1}{d} \pmod 5$ and $d^9 \equiv d \pmod 5$ when d is prime to 5.