A CONGRUENCE PROPERTY OF RAMANUJAN'S FUNCTION $\tau(n)$

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(Communicated by Sir S. S. Bhatnagar, F.R.S.)

(*Received September 3 ; read November 22, 1946.*)

We have recently proved that [1]

(A) $\tau(n) \equiv \sigma(n)$ (mod. 3)

if
$$(n, 3) = 1,$$
where
$$\sum_1^\infty \tau(n)\, x^n = x \prod_1^\infty (1-x^n)^{24}, \ \left[\,|x| < 1\right]$$
and
$$\sigma_k(n) = \sum_{d/n} d^k \text{ and } \sigma_1(n) = \sigma(n).$$
We give below a new and short proof of (A).

§ 1. Case I, n odd.

Defining $\gamma_k\,(n)$ as the no. of integer solutions of
$$n = x_1{}^2 + x_2{}^2 + \ldots\ldots + x_k{}^2$$
Ramanujan [see *Ramanujan* by Hardy, pp. 155] proved
$$r_{24}(n) = \frac{16}{691}\,\sigma_{11}{}^*\,(n) + \frac{128}{691}\left[(-1)^{n-1}\,259\,\tau(n) - 512\,\tau\left(\frac{n}{2}\right)\right]$$
where $\sigma_{11}{}^*\,(n) = \sigma_{11}(n)$ when n is odd.

So that when n is odd

(1) $$691\, r_{24}(n) = 16\,\sigma_{11}(n) + 128.259\,\tau(n).$$
Now

(2) $$(1+2q+2q^4+2q^9+\ldots.)^{24} = 1 + \sum_1^\infty r_{24}(n) q^n.$$

Also

(3) $$\{(1+2q+2q^4+2q^9+\ldots.)^8\}^3 = \left\{1 + \sum_1^\infty c_n q^n\right\}^3 = 1 + \sum_1^\infty e_n q^n$$

where c_n and e_n are integers and $e_n \equiv 0$ (mod 3) when $(n, 3) = 1$.

Therefore from (2) and (3) we have

(4) $$r_{24}(n) \equiv 0 \text{ (mod. 3) when } (n, 3) = 1.$$

[1] Jointly with Gupta and Lahiri in a paper sent to the *Quarterly Journal of Maths.*, Oxford.

Therefore from (1) and (4) we get if $(n, 3) = 1$ and n is odd

$$16\,\sigma_{11}(n)+128.259\tau(n) \equiv 0 \pmod 3$$

or

$$\tau(n) \equiv \sigma_{11}(n) \pmod 3$$
$$\equiv \sigma(n) \pmod 3.$$

§ 2. Case II, $n = 2^m$, $m \geqslant 0$.

We know that [1]

$$\tau(p^\lambda) = \tau(p)\tau(p^{\lambda-1})-p^{11}\tau(p^{\lambda-2})$$

Therefore

$$\tau(2^m) = \tau(2)\tau(2^{m-1})-2^{11}\tau(2^{m-2})$$
$$= -24\tau(2^{m-1})-(3-1)^{11}\tau(2^{m-2})$$
$$\equiv \tau(2^{m-2}) \pmod 3.$$

Also

$$\sigma(2^m) \equiv \sigma(2^{m-2}) \pmod 3$$

for

$$d^3 \equiv d \pmod 3.$$

Therefore

$$\text{if} \quad \tau(2^{m-2}) \equiv \sigma(2^{m-2}) \pmod 3$$
$$\tau(2^m) \equiv \sigma(2^m) \pmod 3.$$

But

$$\tau(2^0) \equiv \sigma(2^0) \pmod 3$$

and

$$\tau(2^1) \equiv \sigma(2^1) \pmod 3.$$

Therefore

$$\tau(2^m) \equiv \sigma(2^m) \pmod 3.$$

§ 3. Case III, n is any even no.

i.e.

$$n = 2^m k,\ m \geqslant 1,\ k \text{ is odd and } (k, 3) = 1$$
$$\tau(2^m\,.\,k) = \tau(2^m)\tau(k)\ \{\text{for } (2^m, k) = 1\}$$

and

$$\sigma(2^m\,.\,k) = \sigma(2^m)\,\sigma(k).$$

Also

$$\tau(2^m) \equiv \sigma(2^m) \pmod 3$$
$$\tau(k) \equiv \sigma(k) \pmod 3.$$

Therefore

$$\tau(2^m\,.\,k) \equiv \sigma(2^m k) \pmod 3.$$

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[1] See *Ramanujan* by Hardy (page 163), Cambridge (1940).