ON TWO PHASE CONFIGURATION OF SMALL MASSES.

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ABSTRACT.

The paper consists of two parts. In Part I physical characteristics of small masses (·2⊙, ·1⊙, ·02⊙, ·01⊙) are calculated for arbitrary values of L/M on the theory of Milne of the two phase configuration of the generalised standard model. The calculations do afford some insight as to the conditions in the interior of such small masses. In Part II we have studied these masses under conditions of incipient degeneracy.

INTRODUCTION.

The theory of stellar masses composed of completely degenerate electron gas has been worked out in complete by Chandrasekhar (1939), who isolated the existence of a limiting mass $M_3 = 5{\cdot}75\mu_e^{-2}$ ⊙. For masses exceeding the limit M_3 the theory predicts no lower limit to the radius nor upper bound to the density. Kothari (1936) has incorporated in the usual white-dwarf the theory of pressure ionisation and has predicted a maximum radius for a cold body. The present paper deals with an investigation of the physical characteristics of masses lying in the region of stellar and proper planetary masses. In Part I we have worked out the properties of small masses on the theory of Milne (1932) of two phase configuration of the generalised standard model. Milne's theory will, however, be a very crude approximation for such masses since the transition region between the degenerate core and the perfect gas region may be quite extensive. At present we shall, however, be satisfied with our approximate calculations, which have been performed for masses, viz. ·2⊙, ·1⊙, ·02⊙ and ·01⊙. Part II deals with these model stars under conditions of incipient degeneracy. We have used Morse's (1940) calculations for the opacity coefficient.

§2. NUMERICAL CALCULATIONS.

The various physical properties given in the present paper are calculated from Milne's formulae given in Kothari's (1932) paper. Since the masses considered here are all less than the critical mass of Milne's theory the configurations are of collapsed type. The tables given below may be considered as an extension of the table given in Kothari's (1932) paper for the model white-dwarf (·5⊙).

TABLE I.

$M = \cdot 2\odot$ $L/M = 10^{-2}$

$C^{-\frac{1}{2}} \equiv \omega_3$	$1-\beta_1 = \frac{k_1 L}{4\pi c G M}$	$\frac{a}{\xi_0} \cdot \frac{\text{Core rad.}}{\text{Total rad.}}$	R_1 Total rad. cm.	ρ_c Central density gm./cm.^{3}	T' Interfacial temperature	T_c Central temp. neglecting conduction	T_c * Central temp. taking account of conduction
∞	0	1	$1\cdot38.\ 10^{9}$	$2\cdot22.\ 10^{5}$	0	$5\cdot10.\ 10^{7}$	$8\cdot15.\ 10^{6}$
20·18	$2\cdot42.\ 10^{-5}$	0·942	$1\cdot47.\ 10^{9}$	$2\cdot22.\ 10^{5}$	$7\cdot10.\ 10^{6}$	$5\cdot10.\ 10^{7}$	$8\cdot71.\ 10^{6}$
8·07	$1\cdot52.\ 10^{-4}$	0·83	$1\cdot48.\ 10^{9}$	$2\cdot14.\ 10^{5}$	$2\cdot41.\ 10^{7}$	$5\cdot10.\ 10^{7}$	$2\cdot50.\ 10^{7}$
4·04	$6\cdot04.\ 10^{-4}$	0·61	$1\cdot73.\ 10^{9}$	$1\cdot92.\ 10^{5}$	$6\cdot06.\ 10^{7}$	$6\cdot40.\ 10^{7}$	$6\cdot08.\ 10^{7}$
2·119	$2\cdot18.\ 10^{-3}$	0·20	$2\cdot90.\ 10^{9}$	$1\cdot47.\ 10^{5}$	$1\cdot43.\ 10^{8}$	$1\cdot43.\ 10^{8}$	$1\cdot43.\ 10^{8}$
2·018	$2\cdot39.\ 10^{-3}$	0	$3\cdot78.\ 10^{9}$	$9\cdot54.\ 10^{4}$	$1\cdot52.\ 10^{8}$	$1\cdot52.\ 10^{8}$	$1\cdot52.\ 10^{8}$

TABLE II.

$M = \cdot 1\odot$ $L/M = 10^{-2}$

$C^{-\frac{1}{2}} \equiv \omega_3$	$1-\beta_1 = \frac{k_1 L}{4\pi c G M}$	$\frac{a}{\xi_0} \cdot \frac{\text{Core rad.}}{\text{Total rad.}}$	R_1 Total rad. cm.	ρ_c Central density gm./cm.^{3}	T' Interfacial temperature	T_c Central temp. neglecting conduction	T_c Central temp. taking account of conduction
∞	0	1	$1\cdot74.\ 10^{9}$	$5\cdot54.\ 10^{4}$	0	$3\cdot46.\ 10^{7}$	$8\cdot45.\ 10^{6}$
20·18	$6\cdot05.\ 10^{-6}$	0·942	$1\cdot86.\ 10^{9}$	$5\cdot54.\ 10^{4}$	$2\cdot82.\ 10^{6}$	$3\cdot46.\ 10^{7}$	$8\cdot45.\ 10^{6}$
8·07	$3\cdot80.\ 10^{-5}$	0·83	$1\cdot86.\ 10^{9}$	$5\cdot38.\ 10^{4}$	$9\cdot58.\ 10^{6}$	$3\cdot46.\ 10^{7}$	$1\cdot14.\ 10^{7}$
4·04	$1\cdot51.\ 10^{-4}$	0·61	$2\cdot17.\ 10^{9}$	$4\cdot79.\ 10^{4}$	$2\cdot41.\ 10^{7}$	$3\cdot50.\ 10^{7}$	$2\cdot46.\ 10^{7}$
2·119	$5\cdot50.\ 10^{-4}$	0·20	$3\cdot65.\ 10^{9}$	$3\cdot80.\ 10^{4}$	$5\cdot69.\ 10^{7}$	$5\cdot70.\ 10^{7}$	$5\cdot70.\ 10^{7}$
2·018	$6\cdot05.\ 10^{-4}$	0	$4\cdot76.\ 10^{9}$	$2\cdot40.\ 10^{4}$	$6\cdot03.\ 10^{7}$	$6\cdot03.\ 10^{7}$	$6\cdot03.\ 10^{7}$

* In the calculation of T_c (with conduction) we have taken account of both radiative and conductive opacity.

The preceding tables for masses $\cdot 2\odot$ and $\cdot 1\odot$ show how the physical quantities vary as we pass from a configuration which is no core to one which is all core for a fixed value of $L/M = 10^{-2}$ ergs./gm. The central density even in a completely degenerate state is not very high.

§3. Having enumerated the various physical quantities for arbitrary values of $\frac{k_1 L}{4\pi c G M}$, we shall now estimate a reasonable value for k_1, the opacity in the gaseous envelope for given L and M. The opacity expression for a Russell mixture of elements undiluted with hydrogen is given by

$$k_1 = 7\cdot34.\ 10^{25} \frac{\rho}{T^{7/2}} \cdot \bar{g}/t. \quad \ldots \quad (1)$$

where $\bar{g}/t$ is the guillotine factor. Let the opacity without the correction factor be given by

$$k_1 = \alpha \cdot \frac{\rho}{T^{7/2}} \cdot \quad \ldots \quad (2)$$

where $\alpha = 7\cdot34.\ 10^{25}.$

The expressions for the interfacial temperature and desnsity are respectively,

$$T' = 8{\cdot}48\,.\,10^9 \left(\frac{k_1 L}{4\pi c G M}\right)^{2/3}. \quad \ldots \quad (3)$$

$$\rho' = D\mu\,.\left(\frac{k_1 L}{4\pi c G M}\right) \quad \ldots \quad (4)$$

where $D = \frac{(R/m_H)^4}{\frac{1}{3}\,aK^3} = 1{\cdot}89.\ 10^7.$

From equations (2), (3), and (4) we have,

$$T' = \left(\frac{D\alpha}{4\pi c G}\right)^{2/7}.\left(\frac{L}{M}\right)^{2/7}. \quad \ldots \quad (5)$$

$$\rho' = \frac{(R/m_H)^{3/2}}{K^{3/2}}.\left(\frac{D\alpha}{4\pi c G}\right)^{3/7}.\left(\frac{L}{M}\right)^{3/7}.\mu \quad \ldots \quad (6)$$

$$k_1 = \frac{(R/m_H)^{3/2}}{K^{3/2}}.\left(\frac{4\pi c G}{D}\right)^{4/7}.\left(\frac{M}{L}\right)^{4/7}.\alpha^{3/7} \quad \ldots \quad (7)$$

The method adopted for determining α is as follows: For given L/M and $\mu = 2{\cdot}1$ and $\alpha = 7{\cdot}34.\ 10^{25}$, as a first approximation T' and ρ' are calculated from (5) and (6). Knowing T' and ρ', $\log_{10} t/\bar{g}$ is known from Morse's table by graphical interpolation. The new value of $\alpha \equiv \alpha_1 = \frac{\alpha}{t/\bar{g}}$ is used to redetermine T' and ρ'. We continue this process of successive approximation till we get coincident values of T' and ρ' for two successive approximations. In practice it was found that generally a third approximation was quite sufficient. The results of such calculations are given below.

For $\frac{L}{M} = 10^{-2}$ ergs/gm.

$T' = 9{\cdot}63.\ 10^6$ degrees.
$\rho' = 7{\cdot}18.\ 10^2$ gm./cm.^{3}
$k_1 = 95{\cdot}43.$
$1-\beta_1 = 3{\cdot}80.\ 10^{-5}.$

§4. Calculations for $M = {\cdot}2\odot$

$\frac{\text{Core radius}}{\text{Total radius}} \approx {\cdot}94$

Central temperature $= 8{\cdot}71.\ 10^6$ degrees.
Central density $= 2{\cdot}22.\ 10^5$ gm./cm.^{3}
Radius $= 1{\cdot}47.\ 10^9$ cm.
Effective temperature $= 7125$ degrees.

§5. Calculations for $M = {\cdot}1\odot$

$\frac{\text{Core radius}}{\text{Total radius}} \approx {\cdot}83$

Central temperature $= 1{\cdot}14.\ 10^7$ degrees.
Central density $= 5{\cdot}38.\ 10^4$ gm./cm.^{3}
Radius $= 1{\cdot}86.\ 10^9$ cm.
Effective temperature $= 5330$ degrees.

§6. We have given before the properties of masses ·2$\odot$ and ·1$\odot$ for $L/M = 10^{-2}$. We shall now do the same for masses ·02$\odot$ and ·01$\odot$ for arbitrary values of L/M $(10^{-4}-10^{-8})$.

TABLE III.

$M = {\cdot}02\odot$

$C^{-\frac{1}{2}} \equiv \omega_3$	$1-\beta_1 = \dfrac{k_1L}{4\pi cGM}$	$\dfrac{a}{\xi_0} \cdot \dfrac{\text{Core rad.}}{\text{Total rad.}}$	R_1 Total rad. cm.	ρ_c Central density gm./cm.^{3}	T' Interfacial temperature.	T_c Central temp. with conduction.			
						$L/M = 10^{-4}$	$L/M = 10^{-5}$	$L/M = 10^{-6}$	$L/M = 10^{-7}$
∞	0	1	$2{\cdot}97.\ 10^{9}$	$2{\cdot}22.\ 10^{3}$	0	$7{\cdot}24.\ 10^{5}$	$3{\cdot}71.\ 10^{5}$	$3{\cdot}39.\ 10^{5}$	$3{\cdot}31.\ 10^{5}$
20·18	$2{\cdot}42.\ 10^{-7}$	0·942	$3{\cdot}18.\ 10^{9}$	$2{\cdot}22.\ 10^{8}$	$3{\cdot}29.\ 10^{5}$	$7{\cdot}24.\ 10^{5}$	$3{\cdot}71.\ 10^{5}$	$3{\cdot}39.\ 10^{5}$	$3{\cdot}31.\ 10^{5}$
8·07	$1{\cdot}52.\ 10^{-6}$	0·83	$3{\cdot}18.\ 10^{9}$	$2{\cdot}14.\ 10^{3}$	$1{\cdot}12.\ 10^{6}$	$1{\cdot}23.\ 10^{6}$	$1{\cdot}15.\ 10^{6}$	$1{\cdot}12.\ 10^{6}$	$1{\cdot}12.\ 10^{6}$
4·04	$6{\cdot}04.\ 10^{-6}$	0·61	$3{\cdot}72.\ 10^{9}$	$1{\cdot}92.\ 10^{3}$	$2{\cdot}81.\ 10^{6}$	$3{\cdot}02.\ 10^{6}$	$2{\cdot}82.\ 10^{6}$	$2{\cdot}82.\ 10^{6}$	$2{\cdot}82.\ 10^{6}$
2·119	$2{\cdot}18.\ 10^{-5}$	0·20	$6{\cdot}24.\ 10^{9}$	$1{\cdot}47.\ 10^{3}$	$3{\cdot}57.\ 10^{6}$	$6{\cdot}62.\ 10^{6}$	$6{\cdot}62.\ 10^{6}$	$6{\cdot}62.\ 10^{6}$	$6{\cdot}62.\ 10^{6}$
2·018	$2{\cdot}39.\ 10^{-5}$	0	$8{\cdot}15.\ 10^{9}$	$9{\cdot}54.\ 10^{2}$	$7{\cdot}06.\ 10^{6}$	$7{\cdot}06.\ 10^{6}$	$7{\cdot}06.\ 10^{6}$	$7{\cdot}06.\ 10^{6}$	$7{\cdot}06.\ 10^{6}$

TABLE IV.

$M = {\cdot}01\odot$

$C^{-\frac{1}{2}} = \omega_3$	$1-\beta_1 = \dfrac{k_1L}{4\pi cGM}$	$\dfrac{a}{\xi_0} \cdot \dfrac{\text{Core rad.}}{\text{Total rad.}}$	R_1 Total rad. cm.	ρ_c Central density gm./cm.^{3}	T' Interfacial temperature	T_c Central temp. with conduction.			
						$L/M = 10^{-5}$	$L/M = 10^{-6}$	$L/M = 10^{-7}$	$L/M = 10^{-8}$
∞	0	1	$3{\cdot}75.\ 10^{9}$	$5{\cdot}54.\ 10^{2}$	0	$2{\cdot}04.\ 10^{5}$	$1{\cdot}38.\ 10^{5}$	$1{\cdot}31.\ 10^{5}$	Same as for $\dfrac{L}{M} = 10^{-7}$.
20·18	$6{\cdot}05.\ 10^{-8}$	0·942	$4{\cdot}00.\ 10^{9}$	$5{\cdot}54.\ 10^{2}$	$1{\cdot}31.\ 10^{5}$	$2{\cdot}04.\ 10^{5}$	$1{\cdot}38.\ 10^{5}$	$1{\cdot}31.\ 10^{5}$	
8·07	$3{\cdot}80.\ 10^{-7}$	0·83	$4{\cdot}01.\ 10^{9}$	$5{\cdot}38.\ 10^{2}$	$4{\cdot}45.\ 10^{5}$	$4{\cdot}68.\ 10^{5}$	$4{\cdot}46.\ 10^{5}$	$4{\cdot}46.\ 10^{5}$	
4·04	$1{\cdot}51.\ 10^{-6}$	0·61	$4{\cdot}68.\ 10^{9}$	$4{\cdot}79.\ 10^{2}$	$1{\cdot}12.\ 10^{6}$	$1{\cdot}12.\ 10^{6}$	$1{\cdot}12.\ 10^{6}$	$1{\cdot}12.\ 10^{6}$	
2·119	$5{\cdot}50.\ 10^{-6}$	0·20	$7{\cdot}85.\ 10^{9}$	$3{\cdot}80.\ 10^{2}$	$2{\cdot}64.\ 10^{6}$	$2{\cdot}64.\ 10^{6}$	$2{\cdot}64.\ 10^{6}$	$2{\cdot}64.\ 10^{6}$	
2·018	$6{\cdot}05.\ 10^{-6}$	0	$1{\cdot}03.\ 10^{10}$	$2{\cdot}40.\ 10^{2}$	$2{\cdot}82.\ 10^{6}$	$2{\cdot}82.\ 10^{6}$	$2{\cdot}82.\ 10^{6}$	$2{\cdot}82.\ 10^{6}$	

§7. The interfacial temperature and density are respectively given by,

Also

$$\left.\begin{aligned} T' &= 4{\cdot}61.\ 10^{7}\left(\frac{L}{M}\right)^{2/7} \\ \rho' &= 1{\cdot}59.\ 10^{4}\left(\frac{L}{M}\right)^{3/7} \\ k_1 &= 10{\cdot}1\left(\frac{M}{L}\right)^{4/7} \\ 1-\beta_1 &= 4{\cdot}01.\ 10^{-4}\left(\frac{L}{M}\right)^{3/7} \end{aligned}\right\} \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad (8)$$

The above expressions are not very accurate since in their deduction we have used ordinary Kramer's formula for the non-degenerate opacity, i.e. $k_1 = 4{\cdot}23.\ 10^{23} \frac{\rho'}{T'^{7/2}}$ instead of the more accurate formula given by Morse.

Having thus estimated a reasonable value for $(1-\beta_1)$ we can now enumerate reasonable values for the physical quantities for the masses $M = {\cdot}02\odot$ and $M = {\cdot}01\odot$.

TABLE V.

$M = {\cdot}02\odot$

L/M	$(1-\beta_1)$	$\frac{a}{\xi_0} \cdot \frac{\text{Core rad.}}{\text{Total rad.}}$	R Total radius cm.	T' Interfacial temperature	ρ' Interfacial density gm./cm.^{3}	k_1 Non-degenerate opacity	T_c Central temperature	ρ_c Central density gm./cm.^{3}	T Effective temperature
10^{-4}	$7{\cdot}74.\ 10^{-6}$	0·57	$3{\cdot}8.\ 10^9$	$3{\cdot}32.\ 10^6$	$3{\cdot}07.\ 10^2$	$1{\cdot}95.\ 10^3$	$3{\cdot}0.\ 10^6$	$1{\cdot}9.\ 10^3$	788
10^{-5}	$2{\cdot}88.\ 10^{-6}$	0·75	$3{\cdot}4.\ 10^9$	$1{\cdot}72.\ 10^6$	$1{\cdot}14.\ 10^2$	$7{\cdot}27.\ 10^3$	$2{\cdot}0.\ 10^6$	$2{\cdot}1.\ 10^3$	468
10^{-6}	$1{\cdot}07.\ 10^{-6}$	0·87	$3{\cdot}2.\ 10^9$	$8{\cdot}90.\ 10^5$	$4{\cdot}26.\ 10^1$	$2{\cdot}72.\ 10^4$	$1{\cdot}2.\ 10^6$	$2{\cdot}2.\ 10^3$	273
10^{-7}	$4{\cdot}01.\ 10^{-7}$	0·95	$3{\cdot}2.\ 10^9$	$4{\cdot}61.\ 10^5$	$1{\cdot}59.\ 10^1$	$1{\cdot}01.\ 10^5$	$3{\cdot}4.\ 10^5$	$2{\cdot}2.\ 10^3$	153

TABLE VI.

$M = {\cdot}01\odot$

L/M	$(1-\beta_1)$	$\frac{a}{\xi_0} \cdot \frac{\text{Core rad.}}{\text{Total rad.}}$	R Total radius cm.	T' Interfacial temperature	ρ' Interfacial density gm./cm.^{3}	k_1 Non-degenerate opacity	T_c Central temperature	ρ_c Central density gm./cm.^{3}	T_e Effective temperature
10^{-5}	$2{\cdot}88.\ 10^{-6}$	0·44	$5{\cdot}2.\ 10^9$	$1{\cdot}72.\ 10^6$	$1{\cdot}14.\ 10^2$	$7{\cdot}27.\ 10^3$	$2{\cdot}0.\ 10^6$	$4{\cdot}2.\ 10^2$	318
10^{-6}	$1{\cdot}07.\ 10^{-6}$	0·68	$4{\cdot}6.\ 10^9$	$8{\cdot}90.\ 10^5$	$4{\cdot}26.\ 10^1$	$2{\cdot}72.\ 10^4$	$1{\cdot}1.\ 10^6$	$4{\cdot}8.\ 10^2$	190
10^{-7}	$4{\cdot}01.\ 10^{-7}$	0·82	$4{\cdot}0.\ 10^9$	$4{\cdot}61.\ 10^5$	$1{\cdot}59.\ 10^1$	$1{\cdot}01.\ 10^5$	$4{\cdot}5.\ 10^6$	$5{\cdot}4.\ 10^2$	115
10^{-8}	$1{\cdot}49.\ 10^{-7}$	0·9	$4{\cdot}0.\ 10^9$	$2{\cdot}39.\ 10^5$	$5{\cdot}92.$	$3{\cdot}75.\ 10^5$	$1{\cdot}3.\ 10^6$	$5{\cdot}5.\ 10^2$	64

§8. Part II. Stars at their maximum Central Temperature and Luminosity.

We have seen in Part I that the configuration in which degeneracy is just setting in has the maximum central temperature. The luminosity of small masses decreases very rapidly with the decrease of mass. It would be worthwhile to calculate the luminosity when the central temperature is maximum. The luminosity L is given by,

$$L = \frac{4\pi c G M (1-\beta_0)}{k_c . \alpha} \quad \ldots\ \ldots\ \ldots\ \ldots \quad (9)$$

where $1-\beta_0$ is given by the quartic equation,

$$1-\beta_0 = 6{\cdot}00.\ 10^{-2} \cdot \left(\frac{M}{\odot}\right)^2 \left(\frac{\mu}{2{\cdot}1}\right)^4 \cdot \beta_0^4, \quad \ldots\ \ldots\ \ldots \quad (10)$$

In our calculations we have assumed the star to be composed of Russell mixture of elements diluted with 35% hydrogen and have taken μ to be equal to unity. This is admittedly a crude assumption. For opacity we have used the expression given by Morse. The guillotine factor is known from Morse's table corresponding to the mean condition $T = \frac{2}{3} T_c$. The results of such calculations are summarised in the table below.

TABLE VII.

Mass $M/\odot$	$\text{Log}_{10} \frac{R}{R\odot}$	T_c Central temperature	ρ_c Central density	$\text{Log}_{10} k_c$.	T_e Effective temperature	$\text{Log}_{10} \frac{L}{L\odot}$ (Calculated)	$\text{Log}_{10} \frac{L}{L\odot}$ (Observed)
·25	$-0·76$	$7·46.\ 10^7$	$3·68.\ 10^3$	0·76	6360	$-1·35$	$-1·77$ Krüger 60
·2	$-0·73$	$2·13.\ 10^7$	$2·36.\ 10^3$	1·12	4217	$-1·99$	$-1·96$ 0_2 Eri C.
·1	$-0·63$	$8·44.\ 10^6$	$5·89.\ 10^2$	2·12	1253	$-3·90$	
·02	$-0·39$	$9·87.\ 10^5$	$2·36.\ 10^1$	4·0	97	$-7·9$	
·01	$-0·29$	$3·92.\ 10^5$	5·89	5·1	27	$-9·9$	

The calculations given in the table are based on the assumption that the central temperature reaches maximum when the core just vanishes, i.e. for $w_3 = 0$. As a matter of fact the maximum will reach somewhere in the region of partial degeneracy. We shall discuss this point in another paper, dealing with partially degenerate stars. Prof. Russell (1944) has performed a similar calculation based on Eddington's idea that high maxima of surface and central temperature should occur for values of the radius about 3 or 4 times that of the degenerate state.

It is clear from the above table that even under the most favourable condition it would be impossible to observe stars of mass less than ·05$\odot$ by their own light. The opacity as calculated from Morse's accurate expression is so large for these small masses that the radiative flux of heat is very small for them.

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