ON STATIONARY LINE-ELEMENTS.

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Abstract.

A set of necessary and sufficient conditions for a spherically symmetrical line-element to be stationary is obtained and a method is given of transforming it to a static form when the conditions are satisfied. A new proof of Birkhoff's theorem is given.

1. Introduction.

Consider a static line-element, $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$.

This may be transformed to a non-static form by an arbitrary non-singular transformation of co-ordinates. Line-elements obtained in this way may be called stationary as is the usual practice. It is the object of this paper to investigate the necessary and sufficient conditions for a non-static line-element to be stationary. Only spherically symmetrical line-elements have been considered.

2. Necessary Conditions.

A general spherically symmetrical static line-element is given by

$$ds^2 = -A\,dr^2 - B(d\theta^2 + \sin^2\theta\, d\phi^2) + D\,dt^2 + 2C\,dr\,dt, \quad \ldots \quad (1)$$

where A, B, C, D are functions of r alone. By the successive transformations

$$\bar{r}^2 = B, \quad \bar{t} = t \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

and

$$\left.\begin{aligned} \bar{r} &= \rho, \\ \bar{t} &= \int -\frac{C}{D}\, d\rho + k\tau, \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

we get

$$ds^2 = -\bar{A}\,d\rho^2 - \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \bar{D}\,d\tau^2, \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where $\bar{A}$ and $\bar{D}$ are functions of ρ alone.

Hence there is no loss of generality in taking the general spherically symmetrical line-element in the form (3).

To get the necessary conditions let it be assumed that the line-element,

$$ds^2 = -A\,dr^2 - B(d\theta^2 + \sin^2\theta\, d\phi^2) + D\,dt^2 + 2C\,dr\,dt,$$

where A, B, C, D are functions of r and t, is transformable into

$$ds^2 = -\bar{A}\,d\rho^2 - \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \bar{D}\,d\tau^2 \quad \ldots \quad \ldots \quad (4)$$

where $\bar{A}$ and $\bar{D}$ are functions of ρ alone.

The law of transformation of tensors gives—

$$\bar{A} = A\left(\frac{\partial r}{\partial \rho}\right)^2 - D\left(\frac{\partial t}{\partial \rho}\right)^2 - 2C\frac{\partial r}{\partial \rho}\cdot\frac{\partial t}{\partial \rho}, \qquad (5.1)$$

$$\rho^2 = B, \qquad (5.2)$$

$$\bar{D} = D\left(\frac{\partial t}{\partial \tau}\right)^2 - A\left(\frac{\partial r}{\partial \tau}\right)^2 + 2C\frac{\partial r}{\partial \tau}\cdot\frac{\partial t}{\partial \tau}, \qquad (5.3)$$

$$0 = C\left(\frac{\partial r}{\partial \rho}\cdot\frac{\partial t}{\partial \tau} + \frac{\partial r}{\partial \tau}\cdot\frac{\partial t}{\partial \rho}\right) + D\frac{\partial t}{\partial \rho}\cdot\frac{\partial t}{\partial \tau} - A\frac{\partial r}{\partial \rho}\cdot\frac{\partial r}{\partial \tau}. \qquad (5.4)$$

Differentiating (5.2) with respect to ρ and τ respectively we get

$$1 = \frac{B'}{2B^{1/2}}\frac{\partial r}{\partial \rho} + \frac{\dot{B}}{2B^{1/2}}\frac{\partial t}{\partial \rho}, \qquad (5.5)$$

$$0 = B'\frac{\partial r}{\partial \tau} + \dot{B}\frac{\partial t}{\partial \tau}, \qquad (5.6)$$

a dot representing a differentiation with regard to t and a dash representing a differentiation with regard to r.

Solving (5.3) to (5.6) we get

$$\frac{\partial r}{\partial \rho} = M, \quad \frac{\partial t}{\partial \rho} = N, \quad \frac{\partial r}{\partial \tau} = R\bar{F}, \quad \frac{\partial t}{\partial \tau} = S\bar{F}, \qquad (6.1)$$

where

$$M = (DB' - C\dot{B}) \,.\, 2B^{1/2}(DB'^2 - 2C\dot{B}B' - A\dot{B}^2)^{-1} \qquad (6.2)$$

$$N = -(A\dot{B} + CB') \,.\, 2B^{1/2}(DB'^2 - 2C\dot{B}B' - A\dot{B}^2)^{-1} \qquad (6.3)$$

$$R = \dot{B}(DB'^2 - 2C\dot{B}B' - A\dot{B}^2)^{-\frac{1}{2}} \qquad (6.4)$$

$$S = -B'(DB'^2 - 2C\dot{B}B' - A\dot{B}^2)^{-\frac{1}{2}}, \qquad (6.5)$$

$$\bar{F} = \bar{D}^{\,1/2}. \qquad (6.6)$$

Differentiating $\frac{\partial r}{\partial \rho}$ with respect to τ and $\frac{\partial r}{\partial \tau}$ with respect to ρ and equating we obtain

$$[(RM' + \dot{M}S) - (R'M + \dot{R}N)]\,\bar{F} - R\,\frac{d\bar{F}}{d\rho} = 0. \qquad (7.1)$$

Similarly, differentiating $\frac{\partial t}{\partial \tau}$ with respect to ρ and $\frac{\partial t}{\partial \rho}$ with respect to τ and equating we obtain

$$[(RN' + \dot{N}S) - (S'M + \dot{S}N)]\,\bar{F} - S\,\frac{d\bar{F}}{d\rho} = 0. \qquad (7.2)$$

Calculation shows that

$$\frac{(RM' + \dot{M}S) - (R'M + \dot{R}N)}{R} = \frac{(RN' + \dot{N}S) - (S'M + \dot{S}N)}{S}$$

$$= \frac{B^{\frac{1}{2}}}{(mB' - n\dot{B})^2}\,[(-m^2B'' + 2mn\dot{B}' - n^2\ddot{B}) + B'(mm' + n\dot{m} - 2\dot{n}m) + \dot{B}(n\dot{n} + n'm - 2nm')], \qquad (8.1)$$

where

$$m = (DB' - C\dot{B}),\ n = (CB' + A\dot{B}) \qquad (8.2)$$

Thus (7.1) and (7.2) reduce to one independent equation. From (7.1) we obtain

$$\frac{\frac{d\bar{F}}{d\rho}}{\bar{F}} = \frac{(RM' + \dot{M}S) - (R'M + \dot{R}N)}{R}. \qquad (9)$$

Since $\frac{d\bar{F}}{d\rho} \Big/ \bar{F}$ is by hypothesis a function of ρ alone, it follows from (9) and (5.2) that

$$\frac{(RM' + \dot{M}S) - (R'M + \dot{R}N)}{R} = \alpha\,(B^{\frac{1}{2}}), \qquad (10.1)$$

where α is arbitrary. Thus we have obtained one of the necessary conditions. By substituting the values of

$$\frac{\partial r}{\partial \rho},\ \frac{\partial t}{\partial \rho}$$

from (6.1) in (5.1) we obtain

$$\bar{A} = AM^2 - 2CMN - DN^2. \qquad (11)$$

Since by hypothesis $\bar{A}$ is a function of ρ alone we obtain from (11) and (5.2)

$$AM^2 - 2CMN - DN^2 = \beta(B^{\frac{1}{2}}) \qquad (10.2)$$

as the second necessary condition. (10.1) and (10.2) is a set of necessary conditions (Vaidya, 1945).

3. Sufficiency of Necessary Conditions.

The conditions obtained in the last section will now be shown to be sufficient. For the line-element,

$$ds^2 = -A\,dr^2 - B(d\theta^2 + \sin^2\theta\, d\phi^2) + D\,dt^2 + 2C\,dr\,dt,$$

where A, B, C, D are functions of r and t, let it be assumed that

$$\frac{(RM' + \dot{M}S) - (R'M + \dot{R}N)}{R} = \alpha(B^{\frac{1}{2}}) \qquad (13.1)$$

and,

$$AM^2 - 2CMN - DN^2 = \beta(B^{\frac{1}{2}}),$$

where M, N, R, S have been defined by (6.2) to (6.6). Define a function $\bar{F}(x)$ by

$$\frac{d\bar{F}}{dx} \Big/ \bar{F} = \alpha(x). \qquad (14)$$

Next consider a transformation defined by

$$\frac{\partial r}{\partial \rho} = M\,;\ \frac{\partial t}{\partial \rho} = N\,;\ \frac{\partial r}{\partial \tau} = R\bar{F}(B^{\frac{1}{2}})\,;\ \frac{\partial t}{\partial \tau} = S\bar{F}(B^{\frac{1}{2}}). \qquad (15)$$

It can be easily verified that

$$\frac{B'}{2B^{\frac{1}{2}}} \cdot \frac{\partial r}{\partial \rho} + \frac{\dot{B}}{2B^{\frac{1}{2}}} \cdot \frac{\partial t}{\partial \rho} = 1, \qquad (16.1)$$

$$B' \frac{\partial r}{\partial \tau} + \dot{B} \frac{\partial t}{\partial \tau} = 0\,; \qquad (16.2)$$

these give

$$\rho^2 = B. \qquad \ldots \qquad (17)$$

The transformation is consistent as can be verified by differentiating $\frac{\partial r}{\partial \rho} = M$ with respect to τ and $\frac{\partial r}{\partial \tau} = N$ with respect to ρ and equating the two values of $\frac{\partial^2 r}{\partial \rho \partial \tau}$ so obtained. This equality is ensured by our definition of $\overline{F}$. The law of transformation of tensors gives the transformed line-element as

$$ds^2 = -\beta(\rho)d\rho^2 - \rho^2(d\theta^2 + \sin^2 \theta d\phi^2) + (\overline{F})^2 d\tau^2, \qquad \ldots \qquad (18)$$

where $\overline{F}$ is given by (14). This is clearly seen to be a static line-element. Thus we have shown that the conditions are sufficient.

4. Proof of Birkhoff's Theorem.

Consider a line-element

$$ds^2 = -A\,dr^2 - B(d\theta^2 + \sin^2\theta d\phi^2) + D dt^2, \qquad \ldots \qquad (19)$$

where A, B, D are functions of r and t and which is a solution of Einstein's field equations for empty space. Hence we have

$$T_\nu^\mu = 0. \qquad \ldots \qquad (20)$$

T_1^1, T_4^1, T_4^4 equated to zero give (Tolman, 1934*a*)

$$\ddot{B} = BD\left[\frac{1}{2}\frac{\dot{B}\dot{D}}{BD^2} + \frac{1}{4}\frac{\dot{B}^2}{DB^2} + \frac{1}{4}\frac{B'^2}{AB^2} + \frac{1}{2}\frac{B'D'}{ABD} - \frac{1}{B}\right] \qquad \ldots \qquad (21.1)$$

$$\dot{B}' = \frac{B}{2}\left[\frac{\dot{B}B'}{B^2} + \frac{\dot{A}B'}{AB} + \frac{D'\dot{B}}{DB}\right], \qquad \ldots \qquad (21.2)$$

$$B'' = AB\left[\frac{1}{4}\frac{B'^2}{AB^2} + \frac{1}{2}\frac{B'A'}{BA^2} + \frac{1}{2}\frac{\dot{A}\dot{B}}{ABD} + \frac{1}{4}\frac{\dot{B}^2}{B^2D} + \frac{1}{B}\right]. \qquad \ldots \qquad (21.3)$$

The necessary and sufficient conditions become for this line-element,

$$\frac{B^{\frac{1}{2}}}{(DB'^2 - A\dot{B}^2)^2}\Big[-2AD\{B''\dot{B}^2 - 2\dot{B}'\dot{B}B' + \ddot{B}.B'^2\}$$

$$+B'\{B'^2DD' + \dot{B}B'(A\dot{D} - 2\dot{A}D)\}$$

$$+\dot{B}\{\dot{B}B'(A'D - 2AD') + \dot{B}^2A\dot{A}\}\Big] = \alpha(B^{\frac{1}{2}}), \qquad \ldots \qquad (22.1)$$

$$\frac{4ADB\ '}{DB'^2 - A\dot{B}^2} = \beta(B^{\frac{1}{2}}). \qquad \ldots \qquad (22.2)$$

On substituting the values of B'', $\dot{B}'$ and $\ddot{B}$ given by equations (21) in the following:

$$\frac{\partial}{\partial r}\left[\frac{4ADB}{DB'^2 - A\dot{B}^2}\right]\dot{B} - \frac{\partial}{\partial t}\left[\frac{4ADB}{DB'^2 - A\dot{B}^2}\right]B' = 0, \qquad \ldots \qquad (23)$$

we find that it becomes an identity. Hence (22.2) is satisfied. On substituting the values of B'', $\dot{B}'$, $\ddot{B}$ in the left-hand side of (22.1) we find that it reduces to

$$-\frac{B^{\frac{1}{2}}}{2B}\left(1-\frac{4ADB}{DB'^2-A\dot{B}^2}\right). \qquad \ldots \quad \ldots \quad \ldots \quad (24)$$

Hence (22.1) is also satisfied.

Thus the necessary and sufficient conditions being satisfied all solutions of Einstein's field-equations for empty space which are spherically symmetrical and which are of the form

$$ds^2 = -A\,dr^2 - B(d\theta^2 + \sin^2\theta d\phi^2) + Ddt^2,$$

are stationary. Thus another proof of Birkhoff's theorem has been given (Tolman, 1934*b*, Einstein and Straus, 1946).

5. Concluding Remarks.

In the course of some recent investigation it was necessary for us to verify whether the following line-element is stationary :

$$ds^2 = -(A+Br^2)^{-2}(dx^2+dy^2+dz^2) - \frac{(\dot{A}+\dot{B}r^2)^2}{(A+Br^2)^2}\cdot\frac{1}{4AB}\,dt^2, \qquad \ldots \quad (25$$

where A and B are arbitrary functions of t. The above set of conditions (10) enabled us to show that it is stationary and the transform (3) for this reveals that it represents flat space-time. The results obtained here should be of use in investigations of spherical distributions where it is necessary to know whether a non-static line-element is stationary and if so to what static form it is transformable.

References.

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