ON THE SIGN OF THE GAUSSIAN SUM.

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It is a classical result due to van der Corput that

$$\int_a^b e^{2\pi i f(x)}\, dx - \sum_{a\leqslant n\leqslant b} e^{2\pi i f(n)} = \tfrac{9}{4}\,\theta_1$$

where $f(x)$ is real, $f'(x)$ monotonic and $|f'(x)| \leqslant \frac{1}{2}$ in (a, b) ; $\theta_1, \theta_2, \theta_3, \ldots$ denote complex numbers whose absolute value does not exceed 1. Hence

$$\sum_0^{k-1} e^{\frac{\pi i m^2}{2k}} - \int_0^{k-1} e^{\frac{\pi i x^2}{2k}}\, dx = \tfrac{9}{4}\,\theta_2 \tag{2}$$

Further as pointed out by Estermann it is trivial that for odd k

$$S = \sum_{m=0}^{k-1} e^{\frac{2\pi i m^2}{k}} = 1 + \frac{2}{(1+i^k)} \sum_{m=1}^{k-1} e^{\frac{\pi i m^2}{2k}} \tag{3}$$

$$\tfrac{1}{2}(1-i)(1+i^k)S = \pm\sqrt{k}. \tag{4}$$

Again by the second mean-value theorem (or graphically), for odd $k \geqslant 13$,

$$\int_{k-1}^{\infty} e^{\frac{\pi i x^2}{2k}}\, dx = \sqrt{k}\int_{\frac{(k-1)^2}{k}}^{\infty} \frac{e^{\frac{\pi i t}{2}}\, dt}{2\sqrt{t}} = \frac{k}{2(k-1)}\,\frac{2\sqrt{2}.2\theta_3}{\pi} \tag{5}$$

$$= \frac{2{\cdot}84}{3{\cdot}1}\,\frac{k}{(k-1)}\,\theta_4 = \theta_5$$

From (2), (3), (5)

$$S\tfrac{1}{2}(1-i)(1+i^k) = \theta_6 + (1-i)\left(\sum_0^{k-1} e^{\frac{\pi i m^2}{2k}} - 1\right)$$

$$= \theta_6 + (\tfrac{9}{4}+1)\sqrt{2}\theta_7 + (1-i)\int_0^{k-1} e^{\frac{\pi i x^2}{2k}}\, dx$$

$$= \theta_6 + \tfrac{17}{4}\sqrt{2}\theta_8 + (1-i)\int_0^{\infty} e^{\frac{\pi i x^2}{2k}}\, dx \tag{6}$$

Now, using (4),

$$\pm\sqrt{k} = (1-i)\sqrt{k}\int_0^\infty e^{\frac{\pi i t^2}{2}}\,dt + \theta_6 + \frac{17\sqrt{2}}{4}\,\theta_8$$

(7) $$= \sqrt{k}\int_0^\infty \left(\cos\frac{\pi x^2}{2} + \sin\frac{\pi x^2}{2}\right)dx + \sqrt{k}i\int_0^\infty \left(\sin\frac{\pi x^2}{2} - \cos\frac{\pi x^2}{2}\right)dx + \left(1+\frac{17\sqrt{2}}{4}\right)\theta_9$$

Making $k \to \infty$, it follows that

(8) $$\int_0^\infty \sin\frac{\pi x^2}{2}\,dx = \int_0^\infty \cos\frac{\pi x^2}{2}\,dx,$$

so that (7) becomes

(9) $$\pm\sqrt{k} = 2\sqrt{k}\int_0^\infty \sin\frac{\pi x^2}{2}\,dx + \left(1+\frac{17\sqrt{2}}{4}\right)\theta_9.$$

Now, since (graphically)

$$\int_0^\infty \sin\left(\frac{\pi x^2}{2}\right)dx = \int_0^\infty \frac{\sin\left(\frac{\pi y}{2}\right)}{2\sqrt{y}}\,dy > 0$$

it follows from (9) that (again by making $k \to \infty$)

$$\int_0^\infty \sin\left(\frac{\pi x^2}{2}\right)dx = \frac{1}{2},$$

so that

(10) $$\pm\sqrt{k} = \sqrt{k} + \left(1+\frac{17\sqrt{2}}{4}\right)\theta_9$$

Hence the + sign holds in (10) if

$$k \geqslant \left(1+\frac{17\sqrt{2}}{4}\right)^2, \text{ i.e. if } k > 49.$$

We have thus proved that the + sign holds in (4) whenever k (which is odd) >49. By actual calculation we can show that the + sign also holds when $k \leqslant 49$.

References.

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