ON THE SELF-ENERGY OF THE ELECTRONS.

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SUMMARY.

The self-energy of an electron in motion is investigated from quantum electrodynamics. In the first article the interaction energy of an electron with electromagnetic field is given and in the second and third articles the evaluation of the dynamic and static self-energies is undertaken. The divergence of the self-energy in the hole theory is logarithmic which is shown to be due to the symmetrisation in the behaviour of the electron with respect to emission and absorption of photon in its initial state.

INTRODUCTION.

The self-energy of an electron is its total energy in free space when isolated from other particles. In the classical theory of Lorentz where the electron is considered to have a finite extension over which the total charge is distributed the self-energy includes besides the energy of its mass also the interaction energy between the different charge elements of the electron integrated over its volume, the interaction taking place via the electromagnetic field produced at each point of the electron by its other parts. In the first approximation when the retardation of the electromagnetic field inside the electron is neglected we obtain the usual electrostatic self-energy, i.e., the energy which is required to keep the different parts of the charges of the electron together and this tends to infinity in the limit for a point charge. In the higher approximations the self-energy is due to the self-force which arises because of the retardation of the electromagnetic field inside the electron and contains terms the first of which is independent of the structure of the electron and is the usual radiation damping; the second and the higher approximation terms on the other hand can be expressed, particularly for the case of a simple oscillator, as a series in powers of r_0/λ where r_0 is the radius of the electron and λ the wave length of the emitted radiation. This part of the self-energy therefore vanishes for a point charge. In the quantum theory where the electron is considered to be a point charge the self-energy which has been calculated by Waller (1930) Oppenheimer (1930) and Rosenfeld (1931) exhibits also the strong divergent character tending to infinity. It has been therefore argued from the idea of the correspondence principle that this infinite self-energy is due to our assumption of a point electron. But it has been already emphasised by Bhabha and Corben (1941) that the infinite self-energy in the theory of Lorentz lies in his assumption that a point electron can be considered as the limit of a finite electron in which the charge is distributed, the work which is required in compressing a finite charge distributed over a finite volume into an infinitely small volume against the electromagnetic forces being infinite. Dirac (1938) and Pryce (1938) and more recently Bhabha and Harish-Chandra (1946) have shown that it is quite possible to construct a scheme of point electron in the classical theory from which the infinite self-energy can be eliminated in a relativistic invariant way. But unfortunately the scheme in some cases leads to solutions which are in conflict with the physical ideas (Eliezer, 1943). It has however not been possible as yet to develop an analogous scheme of Dirac for removing in a relativistic invariant way the infinites which appear in the equation characterising the electron in quantum mechanics in spite of the fact that it is treated as a point. Dirac's (1942) new method of field quantisation by introducing the λ-limiting process and the negative energy photons to ensure the convergence has also its validity within a limited scope. The λ-limiting process fails altogether in the hole theory which alone satisfactorily accounts for the creation and annihilation of electron-position pair (Pauli, 1943). It has further the drawback that it does not lead to the radiation damping which is so significant, particularly in the field theory of meson. The discussion of these problems from Dirac's quantum electrodynamics will be taken up in detail in a subsequent paper. It is to be noted that the strong divergent character of self-energy for an electron in quantum mechanics in the original one electron theory of Dirac, in which it is assumed that all the negative energy states are empty, is obviously due to the emission of very high energy photons in the intermediate states and lies more in our assumption that the electron in the initial state can only emit but not absorb the photon. But the self-energy in the hole theory is given by the difference of the self-energy of the electron in the positive energy when the negative energy states are occupied and that of the vacuum electrons filling up all the negative energy states. The introduction of vacuum's contribution is just equivalent to allowing the electron also to absorb the photon in the initial state and it is this symmetrisation in the behaviour of the electron that reduces the self-energy to diverge only logarithmically. In the present paper we shall give a straight-forward calculation of self-energies for a moving electron from quantum electrodynamics. The calculations of self-energy were also undertaken by Weisskopf (1934) by following closely the method of classical electrodynamics; his results, however, remained incomplete due to some oversights in the calculations.

1. *Interaction energy of the electron with the electromagnetic field.*

The Lagrangian of the Dirac electron in the electromagnetic field is given as usual by

$$L = i\hbar c\left(\psi^+\gamma^\mu\frac{\partial\psi}{\partial x_\mu}+k\psi^+\psi\right)+e\psi^+\gamma^\mu\phi_\mu\psi-\frac{1}{16\pi}F_{\mu\nu}F_{\mu\nu} \quad .. \quad (1)$$

with

$$\psi^+ = i\psi^*\beta,\ \gamma^k = -i\beta\alpha^k,\ \gamma^4 = \beta,\ k = \frac{mc}{\hbar} \quad .. \quad .. \quad (2)$$

and

$$F_{\mu\nu} = \frac{\partial\phi_\nu}{\partial x_\mu}-\frac{\partial\phi_\mu}{\partial x_\nu} \quad .. \quad .. \quad .. \quad .. \quad (3)$$

where ψ is the wave function of the electron, α, β are well-known Dirac's matrices and ϕ_μ is the 4-vector potential of the field with

$$\phi_k = A_k,\ \phi_4 = iA_0,\ \mu,\nu = 1, 2, 3, 4, \text{ and } k = 1, 2, 3 \quad .. \quad .. \quad (4)$$

Further

$$F_{4k} = iE_k, \quad F_{23},\ F_{31},\ F_{12} = H \quad .. \quad .. \quad .. \quad (5)$$

and

$$\frac{1}{16\pi}F_{\mu\nu}F_{\mu\nu} = \frac{1}{8\pi}(H^2-E^2) \quad .. \quad .. \quad .. \quad (6)$$

$\boldsymbol{E}$ and $\boldsymbol{H}$ being the electric and magnetic fields. It is to be noted that the electromagnetic field here includes besides the field which is produced by the electron itself due to its motion and spin, also the field due to the Zero-point fluctuations of the radiation field.

The Hamiltonian function can be readily obtained from the Lagrangian given by (1)

$$H = \int\left(\frac{\partial L}{\partial\left(\frac{\partial\psi}{\partial x_4}\right)}\cdot\frac{\partial\psi}{\partial x_4}+\frac{\partial L}{\partial\left(\frac{\partial\phi_k}{\partial x_4}\right)}\cdot\frac{\partial\phi_k}{\partial x_4}-L\right)dr \quad .. \quad .. \quad (7)$$

$$= \int \left\{ -i\hbar c\left(\psi^+\gamma^k\frac{\partial\psi}{\partial x_k}+k\psi^+\psi\right)-e\psi^+\gamma^\mu\phi_\mu\psi-\frac{1}{4\pi}F_{4k}\frac{\partial\phi_4}{\partial x_k} -\frac{1}{4\pi}F_{4k}F_{4k}+\frac{1}{16\pi}F_{\mu\nu}F_{\mu\nu}\right\}dr \quad \text{.. (8)}$$

or expressing in terms of electric and magnetic field strengths we obtain

$$H = \int \left[\psi^*\left\{c\alpha\left(p-\frac{e}{c}A\right)+\beta mc^2\right\}\psi+e\psi^*A_0\psi-\frac{1}{4\pi}A_0 \text{ div } E+\frac{1}{8\pi}(E^2+H^2)\right]dr \quad \text{.. (9)}$$

$$= H_e+H_F+H_D+H_S \quad \text{.. .. (10)}$$

where H_e and H_F are the usual Hamiltonian for the electron and the electromagnetic field, and H_D and H_S are the interaction energies of the electron with static and dynamic part of the field, the expectation values of which are the self-energies.

$$H_D = -e\int \psi^*(\alpha A)\psi \, dr \quad \text{.. .. .. (11)}$$

$$H_S = -e\int \{\psi^*(\alpha A_l)\psi-\psi^*A_0\psi\}dr \quad \text{.. .. .. (12)}$$

where A and A_l are the transverse and longitudinal part of the field.

2. *Electrodynamic self-energy.*

We now proceed to calculate the self-energy given by the interaction (11), where

$$A = \sum_k \sqrt{\frac{2\pi c\hbar^2}{k}}\, e(C_k e^{i(kr)/\hbar}+C_k^* e^{-i(kr)/\hbar}) \quad \text{.. .. (13)}$$

and

$$\psi = \sum_p a(p)\, u(p)\, e^{i(pr)/\hbar} \quad \text{.. .. .. (14)}$$

Here C_k and C_k^* are absorption and emission operators decreasing and increasing respectively the number of photons by one. $a(p)$, $a^*(p)$ are similar operators for the electrons. Substituting these values in (11) we obtain for the interaction energy of the electron with the field.

$$H_D = \sum_p \sum_{p'} \sum_k -e\sqrt{\frac{2\pi c\hbar^2}{k}}\{C_k\delta(p-p'+k)+C_k^*\delta(p-p'-k)\} N(p')\Delta^*(p')V(p')V(p)\Delta(p)N(p)\big(u^*(p')\alpha u(p)\big) \quad \text{.. (15)}$$

where $N(p)$ and $N(p')$ give the number of electrons in the states p and p'. $\Delta(p)$, $\Delta(p')$ are operators which operating on $N(p)$, etc., change them to $1-N(p)$ etc., and $V(p)$ and $V(p')$ are the Jordon-Wigner's Vorzeichenfunctions given by $+1$ or -1 according as the number of occupied states arranged in some definite manner before the state referred to is even or odd. Now the expectation value of H_D is given in the first approximation by its diagonal matrix elements which is obviously zero. Thus the self-energy is given by the second approximation of the interaction energy

$$W_D = \sum_m \frac{H_{Am}H_{mA}}{E_A-E_m} \quad \text{.. .. .. .. (16)}$$

where the summation is to be taken over all the intermediate states.

(*a*) Hole Theory:

The self-energy is given by the difference of the contributions due to the following transitions of the electron in the positive energy state and the vacuum electrons.

I. The electron $u(\boldsymbol{p}_0)$ emits a photon $\boldsymbol{k}$ in going over to the state $u(\boldsymbol{p}_0-\boldsymbol{k})$; it then absorbs the photon $\boldsymbol{k}$ and comes back to the original state $u(\boldsymbol{p}_0)$.

$$E_A - E_I = E(\boldsymbol{p}_0) - ck - E(\boldsymbol{p}_0 - \boldsymbol{k}) \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

II. The vacuum electron $\underline{u}(\boldsymbol{p}_0+\boldsymbol{k})$ emits a photon $\boldsymbol{k}$ in going over to the positive energy state $u(\boldsymbol{p}_0)$; it then absorbs the photon $\boldsymbol{k}$ and comes back to the original negative energy state.

$$E_A - E_{II} = -\big(E(\boldsymbol{p}_0) + ck + E(\boldsymbol{p}_0+\boldsymbol{k})\big) \quad \ldots \quad \ldots \quad (18)$$

The self-energy is thus given by

$$W_D = W_{Vac+1} - W_{Vac}$$

$$= \sum_{\boldsymbol{k}} \frac{2\pi e^2\hbar^2 c}{k} \left\{ \frac{\big(u(\boldsymbol{p}_0)^* \alpha u(\boldsymbol{p}_0-\boldsymbol{k})\big)\big(u(\boldsymbol{p}_0-\boldsymbol{k})^* \alpha u(\boldsymbol{p}_0)\big)}{E(\boldsymbol{p}_0) - ck - E(\boldsymbol{p}_0-\boldsymbol{k})} \right.$$

$$\left. + \frac{\big(u(\boldsymbol{p}_0)^* \alpha \underline{u}(\boldsymbol{p}_0+\boldsymbol{k})\big)\big(\underline{u}(\boldsymbol{p}_0+\boldsymbol{k})^* \alpha u(\boldsymbol{p}_0)\big)}{E(\boldsymbol{p}_0) + ck + E(\boldsymbol{p}_0+\boldsymbol{k})} \right\} \ldots \quad (19)$$

The summation is over the spin directions of the electrons as well as over all sorts of photons in the intermediate states. Now carrying out the summation over the spins in the intermediate states and averaging in the initial states we obtain for the first term in the bracket

$$\frac{1}{8E(\boldsymbol{p}_0-\boldsymbol{k})E(\boldsymbol{p}_0)} Sp\; \alpha\big(E(\boldsymbol{p}_0-\boldsymbol{k}) + H(\boldsymbol{p}_0-\boldsymbol{k})\big)\alpha\big(E(\boldsymbol{p}_0) + H(\boldsymbol{p}_0)\big)$$

$$= \frac{1}{8E(\boldsymbol{p}_0-\boldsymbol{k})E(\boldsymbol{p}_0)} Sp\; \alpha\big(c(\alpha,\ \boldsymbol{p}_0-\boldsymbol{k}) + \beta mc^2 + E(\boldsymbol{p}_0-\boldsymbol{k})\big)\alpha\big(c(\alpha\boldsymbol{p}_0) + \beta mc^2 + E(\boldsymbol{p}_0)\big)$$

$$= \frac{1}{8E(\boldsymbol{p}_0-\boldsymbol{k})E(\boldsymbol{p}_0)} \big(-4c^2 p_0^2 + 8c^2(\boldsymbol{e}\boldsymbol{p}_0)(\boldsymbol{e}\boldsymbol{p}_0) + 4c^2(\boldsymbol{k}\boldsymbol{p}_0) - 4m^2c^4 + 4E(\boldsymbol{p}_0)E(\boldsymbol{p}_0-\boldsymbol{k})\big)$$

Summing over the polarisation of the photons in the intermediate states with the help of the relation

$$\sum_{\boldsymbol{e}} (\boldsymbol{e}\boldsymbol{p})(\boldsymbol{e}\boldsymbol{p}) = (\boldsymbol{p}\boldsymbol{p}) - \frac{(\boldsymbol{p}\boldsymbol{k})(\boldsymbol{p}\boldsymbol{k})}{k^2} \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

the first term becomes

$$\frac{1}{E(\boldsymbol{p}_0)E(\boldsymbol{p}_0-\boldsymbol{k})} \left\{ E(\boldsymbol{p}_0)E(\boldsymbol{p}_0-\boldsymbol{k}) + c^2(\boldsymbol{k}\boldsymbol{p}_0) - \frac{c^2(\boldsymbol{p}_0\boldsymbol{k})(\boldsymbol{p}_0\boldsymbol{k})}{k^2} - m^2c^4 \right\} \ldots \quad (21)$$

Similarly for the second term in the bracket we obtain

$$\frac{1}{8E(\boldsymbol{p}_0)E(\boldsymbol{p}_0+\boldsymbol{k})} Sp\; \alpha\big(-c(\alpha,\ \boldsymbol{p}_0+\boldsymbol{k}) - \beta mc^2 + E(\boldsymbol{p}_0+\boldsymbol{k})\big)\alpha\big(c(\alpha\boldsymbol{p}_0) + \beta mc^2 + E(\boldsymbol{p}_0)\big)$$

$$= \frac{1}{E(\boldsymbol{p}_0)E(\boldsymbol{p}_0+\boldsymbol{k})} \left\{ E(\boldsymbol{p}_0)E(\boldsymbol{p}_0+\boldsymbol{k}) + c^2(\boldsymbol{p}_0\boldsymbol{k}) + c^2\frac{(\boldsymbol{p}_0\boldsymbol{k})(\boldsymbol{p}_0\boldsymbol{k})}{k^2} + m^2c^4 \right\} \ldots \quad \ldots \quad (22)$$

We now replace the summation over all the photons by integration and multiply thereby the expression (2) by $\frac{k^2 dk\, d\Omega}{8\pi^3\hbar^3}$ which denotes the number of photons lying within the solid angle $d\Omega$ and within the range k and $k+dk$. The expression for the self-energy thus becomes

$$W_D = \frac{e^2c}{4\pi^2\hbar}\int (I_1+I_2)k\,dk \qquad \text{(23)}$$

where

$$I_1 = \int d\Omega\left[\frac{E(\mathbf{p}_0)E(\mathbf{p}_0-\mathbf{k})+c^2(\mathbf{p}_0\mathbf{k})-c^2\dfrac{(\mathbf{p}_0\mathbf{k})(\mathbf{p}_0\mathbf{k})}{k^2}-m^2c^4}{E(\mathbf{p}_0)E(\mathbf{p}_0-\mathbf{k})\big(E(\mathbf{p}_0)-ck-E(\mathbf{p}_0-\mathbf{k})\big)}\right] \qquad \text{(24)}$$

$$I_2 = \int d\Omega\left[\frac{E(\mathbf{p}_0)E(\mathbf{p}_0+\mathbf{k})+c^2(\mathbf{p}_0\mathbf{k})+c^2\dfrac{(\mathbf{p}_0\mathbf{k})(\mathbf{p}_0\mathbf{k})}{k^2}+m^2c^4}{E(\mathbf{p}_0)E(\mathbf{p}_0+\mathbf{k})\big(E(\mathbf{p}_0)+ck+E(\mathbf{p}_0+\mathbf{k})\big)}\right] \qquad \text{(25)}$$

On integrating over the direction of $\mathbf{k}$ we have

$$I_1 = \frac{2\pi}{c}\left[-\frac{(E^3+E^2kc+Ek^2c^2+k^3c^3)(E_+-E_-)}{4Ep_0k^3c^3}-\frac{(E-kc)}{Ek}+\frac{(E-kc)(E_+^3-E_-^3)}{12Ep_0k^3c^3}+\frac{m^2c^4}{Ep_0kc}\log\frac{E-kc-E_+}{E-kc-E_-}\right] \qquad \text{(26)}$$

$$I_2 = \frac{2\pi}{c}\left[\frac{(E^3-E^2kc+Ek^2c^2-k^3c^3)(E_+-E_-)}{4Ep_0k^3c^3}+\frac{E+kc}{Ek}-\frac{(E+kc)(E_+^3-E_-^3)}{12Ep_0k^3c^3}+\frac{m^2c^4}{Ep_0kc}\log\frac{E+kc+E_+}{E+kc+E_-}\right] \qquad \text{(27)}$$

where

$$E = c\sqrt{p_0^2+m^2c^2},\quad E_+ = \sqrt{E^2+k^2c^2+2c^2kp_0},\quad E_- = \sqrt{E^2+k^2c^2-2c^2kp_0} \qquad \text{(28)}$$

Whence we obtain finally

$$W_D = \frac{e^2}{2\pi\hbar}\int k\,dk\left\{\frac{-(E^2+k^2c^2)(E_+-E_-)}{2Ep_0k^2c^2}+\frac{2c}{E}-\frac{(E_+^3-E_-^3)}{6Ep_0k^2c^2}+\frac{m^2c^4}{Ep_0kc}\log\frac{E^2-(kc+E_+)^2}{E^2-(kc+E_-)^2}\right\} \qquad \text{(29)}$$

The expression is sufficiently complicated for interpretation. We can, however, evaluate it when the kinetic energy of the electron is small compared with its rest energy. We have then

$$W_D = \frac{e^2}{2\pi\hbar cE}\operatorname*{Lt}_{k\to\infty}\left[c^2k^2-ck\sqrt{E^2+k^2c^2}+m^2c^4\log\frac{kc+\sqrt{E^2+k^2c^2}}{mc^2}-p_0{}^2c^2\left\{\frac{4}{3}\log\frac{kc+\sqrt{E^2+k^2c^2}}{mc^2}+\frac{kc}{\sqrt{E^2+k^2c^2}}-\frac{1}{3}\frac{k^3c^3}{(E^2+k^2c^2)^{3/2}}\right\}\right] \qquad \text{(30)}$$

$$= \frac{e^2}{2\pi\hbar c}mc^2\left(1-\frac{11}{6}\frac{p_0{}^2}{m^2c^2}\right)\operatorname*{Lt}_{k\to\infty}\log\frac{k}{mc}+\text{ finite terms} \qquad \text{(31)}$$

This approximate result has been given also by Weisskopf. The self-energy diverges only logarithmically.

(*b*) Original Theory:

It may be of some interest to deduce the expression for self-energy in the original theory of Dirac where it is assumed that all the negative energy states of the electron are empty. We have the following processes to consider:—

I. The same as before.

II. The electron $u(\boldsymbol{p}_0)$ emits a photon $\boldsymbol{k}$ in going over to the negative energy state $\underline{u}(\boldsymbol{p}_0-\boldsymbol{k})$; it then absorbs the photon $\boldsymbol{k}$ and comes back to the original state $u(\boldsymbol{p}_0)$.

$$E_A-E_{II} = E(p_0)-ck+E(\boldsymbol{p}_0-\boldsymbol{k}) \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

We have thus

$$W_D = \sum_{\boldsymbol{k}} \frac{2\pi e^2\hbar^2 c}{k}\left\{\frac{(u(\boldsymbol{p}_0)^*\alpha u(\boldsymbol{p}_0-\boldsymbol{k}))(u(\boldsymbol{p}_0-\boldsymbol{k})^*\alpha u(\boldsymbol{p}_0))}{E(p_0)-ck-E(\boldsymbol{p}_0-\boldsymbol{k})} + \frac{(u(\boldsymbol{p}_0)^*\alpha \underline{u}(\boldsymbol{p}_0-\boldsymbol{k}))(\underline{u}(\boldsymbol{p}_0-\boldsymbol{k})\alpha u(\boldsymbol{p}_0))}{(Ep_0)-ck+E(\boldsymbol{p}_0-\boldsymbol{k})}\right\} \quad \ldots \quad (33)$$

Now carrying out the summation over the spins and polarisation in the intermediate states and averaging over the spins of the electron in the initial states as before we obtain:—

$$W_D = \frac{e^2c}{4\pi\hbar}\int (I_1+I_2')k\,dk \quad \ldots \quad \ldots \quad \ldots \quad (34)$$

where

$$I_2' = \int d\Omega \left[\frac{E(p_0)E(\boldsymbol{p}_0-\boldsymbol{k})-c^2(\boldsymbol{p}_0\boldsymbol{k})+\dfrac{c^2(\boldsymbol{p}_0\boldsymbol{k})(\boldsymbol{p}_0\boldsymbol{k})}{k^2}+m^2c^4}{E(p_0)E(\boldsymbol{p}_0-\boldsymbol{k})(E(p_0)-ck+E(\boldsymbol{p}_0-\boldsymbol{k}))}\right] \quad \ldots \quad (35)$$

which on integration over the direction of $\boldsymbol{k}$ reduces to

$$I_2' = \frac{2\pi}{c}\left[\frac{(E^3+E^2kc+Ek^2c^2+k^3c^3)(E_+-E_-)}{4Ep_0k^3c^3}-\frac{(E-kc)}{Ek}-\frac{(E-kc)(E_+^3-E_-^3)}{12Ep_0k^3c^3} + \frac{m^2c^4}{Ep_0kc}\log\frac{E-kc+E_+}{E-kc+E_-}\right] \quad \ldots \quad (36)$$

and we obtain

$$W_D = \frac{e^2}{\pi\hbar cE}\left[c^2\int_0^\infty k\,dk+\left(\frac{m^2c^4}{2p_0}\log\frac{E+p_0c}{E-p_0c}-Ec\right)\int_0^\infty dk\right] \quad \ldots \quad (37)$$

This shows that the self-energy diverges. The expression (37) was first obtained by Waller.

3. *Electrostatic self-energy.*

In classical electrodynamics the expression for electrostatic self-energy as given by (12) is evaluated with the help of the well-known Maxwell-Lorentz equations

$$\text{rot}\, \boldsymbol{H}-\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \frac{4\pi}{c}\boldsymbol{i}, \quad \text{div}\, \boldsymbol{E} = 4\pi\rho$$

and,

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \text{grad}\, A_0, \quad \boldsymbol{H} = \text{rot}\, \boldsymbol{A} \quad \ldots \quad \ldots \quad (38)$$

together with

$$\text{div}\, \boldsymbol{E}_{\text{trans}} = 0, \ \text{rot}\, \boldsymbol{E}_l = 0, \ \boldsymbol{H}_l = 0 \quad \ldots \quad \ldots \quad \ldots \quad (39)$$

Now for Dirac electron

$$\rho = e\,(\psi^*\psi),\ \boldsymbol{i} = e\,(\psi^*\alpha\psi) \qquad \ldots \quad \ldots \quad \ldots \quad (40)$$

Therefore (12) gives

$$H_s = \int \left\{ \rho A_0 - \frac{1}{c}(\boldsymbol{i}A_l) \right\} d\boldsymbol{r}$$

$$= \frac{1}{8\pi} \int \left[E_l^2 + \frac{1}{c}\frac{d}{dt}(A_l E) \right] d\boldsymbol{r} \qquad \ldots \quad \ldots \quad \ldots \quad (41)$$

The electrostatic self-energy which is defined as the expectation value of H_s is therefore given by

$$W_s = \overline{H}_s = \frac{1}{8\pi} \int E_l^2\, d\boldsymbol{r} = \frac{1}{2} \int \frac{\rho(\boldsymbol{r})\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\, d\boldsymbol{r}\, d\boldsymbol{r}' \qquad \ldots \quad \ldots \quad (42)$$

For a point electron this becomes infinite. We cannot take over this result directly into quantum mechanics. It is usually left unquantised and subtracted from the Hamiltonian representing the interaction. It may be of some interest, however, to attempt to evaluate the expression (12) exactly in an analogous way as in the transverse part of the self-energy. We accordingly express the quantities A_l and A_0 as

$$A_l = \sum_{\boldsymbol{k}} \sqrt{\frac{2\pi c\hbar^2}{k}} \cdot \boldsymbol{n}\left(c'_{\boldsymbol{k}}\, e^{i(\boldsymbol{kr})/\hbar} + c'^*_{\boldsymbol{k}}\, e^{-i(\boldsymbol{kr})/\hbar}\right) \qquad \ldots \quad \ldots \quad (43)$$

$$A_0 = \sum_{\boldsymbol{k}} \sqrt{\frac{2\pi c\hbar^2}{k}} \left(c'_{\boldsymbol{k}}\, e^{i(\boldsymbol{kr})/\hbar} + c'^*_{\boldsymbol{k}}\, e^{-i(\boldsymbol{kr})/\hbar}\right) \qquad \ldots \quad \ldots \quad (44)$$

where $\boldsymbol{n} = \frac{\boldsymbol{k}}{k}$ is the unit vector in the direction of the wave vector $\boldsymbol{k}$. $c'_{\boldsymbol{k}}$, $c'^*_{\boldsymbol{k}}$ are operators as before denoting respectively the decrease and the increase in the number of longitudinal photons. The interaction energy of the electron with the longitudinal part of the field is obtained from (12) as

$$H_s = \sum_{\boldsymbol{p}} \sum_{\boldsymbol{p}'} \sum_{\boldsymbol{k}} -e \sqrt{\frac{2\pi c\hbar^2}{k}} \left\{ c'_{\boldsymbol{k}}\, \delta(\boldsymbol{p}-\boldsymbol{p}'+\boldsymbol{k}) + c'^*_{\boldsymbol{k}}\, \delta(\boldsymbol{p}-\boldsymbol{p}'-\boldsymbol{k}) \right\}$$

$$N(\boldsymbol{p}')\Delta^*(\boldsymbol{p}')V(\boldsymbol{p}')V(\boldsymbol{p})\Delta(\boldsymbol{p})N(\boldsymbol{p}) \times \left(u(\boldsymbol{p}')^*\alpha_{\boldsymbol{k}}u(\boldsymbol{p}) - u(\boldsymbol{p}')^*u(\boldsymbol{p})\right) \quad \ldots \quad (45)$$

The self-energy is then obtained in an analogous way as before from the expression

$$W_s = \sum_m \frac{H^{sl}_{Am} H^{sl}_{mA}}{E_A - E_m} - \sum_m \frac{H^{so}_{Am} H^{so}_{mA}}{E_A - E_m} \qquad \ldots \quad \ldots \quad \ldots \quad (46)$$

the two parts being the contributions of the longitudinal and the scalar fields respectively.

(*a*) Hole Theory:

Considering the self-energy as the difference between the self-energy when an electron is present in the positive energy state and the self-energy of vacuum we obtain

$$W_s = \sum_{\boldsymbol{k}} \frac{2\pi e^2\hbar^2 c}{k} \left\{ \frac{\left(u(\boldsymbol{p}_0)^*\alpha_{\boldsymbol{k}}u(\boldsymbol{p}_0-\boldsymbol{k})\right)\left(u(\boldsymbol{p}_0-\boldsymbol{k})^*\alpha_{\boldsymbol{k}}u(\boldsymbol{p}_0)\right) - \left(u(\boldsymbol{p}_0)^*u(\boldsymbol{p}_0-\boldsymbol{k})\right)\left(u(\boldsymbol{p}_0-\boldsymbol{k})^*u(\boldsymbol{p}_0)\right)}{E(\boldsymbol{p}_0) - ck - E(\boldsymbol{p}_0-\boldsymbol{k})} \right.$$

$$\left. + \frac{\left(u(\boldsymbol{p}_0)^*\alpha_{\boldsymbol{k}}\underline{u}(\boldsymbol{p}_0+\boldsymbol{k})\right)\left(\underline{u}(\boldsymbol{p}_0+\boldsymbol{k})^*\alpha_{\boldsymbol{k}}u(\boldsymbol{p}_0)\right) - \left(u(\boldsymbol{p}_0)^*\underline{u}(\boldsymbol{p}_0+\boldsymbol{k})\right)\left(\underline{u}(\boldsymbol{p}_0+\boldsymbol{k})u(\boldsymbol{p}_0)\right)}{E(\boldsymbol{p}_0) + ck + E(\boldsymbol{p}_0+\boldsymbol{k})} \right\} (47)$$

where
$$\alpha_k = \frac{(\alpha k)}{k} \quad \ldots \quad \ldots \quad . \quad \ldots \quad (48)$$

Making use of Dirac's wave equation we can write

$$u(p_0)^*(\alpha k)u(p_0-k) = \frac{1}{c}\left[E(p_0)-E(p_0-k)\right]u(p_0)^*u(p_0-k)$$

$$u(p_0)^*(\alpha k)\underline{u}(p_0+k) = -\frac{1}{c}\left[E(p_0+k)+E(p_0)\right]u(p_0)^*\underline{u}(p_0+k), \text{ etc.} \quad (49)$$

The expression (47) thus reduces to

$$W_s = \sum_k \frac{2\pi e^2\hbar^2}{ck^3}\Big\{\left[E(p_0)+ck-E(p_0-k)\right]\big(u(p_0)^*u(p_0-k)\big)\big(u(p_0-k)^*u(p_0)\big)$$
$$+\left[E(p_0)-ck+E(p_0+k)\right]\big(u(p_0)^*\underline{u}(p_0+k)\big)\big(\underline{u}(p_0+k)^*u(p_0)\big)\Big\} \quad \ldots \quad (50)$$

Carrying out the summation and averages as before

$$W_s = \frac{\pi e^2\hbar^2}{4ck^3E(p_0)}Sp\left\{\frac{\big(E(p_0-k)+H(p_0-k)\big)\big(E(p_0)+H(p_0)\big)\big(E(p_0)+ck-E(p_0-k)\big)}{E(p_0-k)}\right.$$
$$\left.+\frac{\big(E(p_0+k)-H(p_0+k)\big)\big(E(p_0)+H(p_0)\big)\big(E(p_0)-ck+E(p_0+k)\big)}{E(p_0+k)}\right\}$$

Now on evaluating the spur and summing over the intermediate states of the photon we obtain finally

$$W_s = \frac{e^2}{8\pi^2\hbar c}\int (I_3+I_4)\frac{dk}{k} \quad \ldots \quad \ldots \quad \ldots \quad (51)$$

where

$$I_3 = \int d\Omega\left[\frac{\big(E(p_0)+ck\big)\big(E(p_0)^2-c^2(p_0k)\big)}{E(p_0)E(p_0-k)} - \frac{\big(E(p_0)-ck\big)\big(E(p_0)^2+c^2(p_0k)\big)}{E(p_0)E(p_0+k)}\right] \ldots \quad (52)$$

$$I_4 = \int d\Omega\, E(p_0)\left[E(p_0+k)-E(p_0-k)\right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (53)$$

The integration over the direction of k gives

$$W_s = \frac{e^2}{4\pi\hbar c}\int_0^\infty\left[\frac{(E^2-k^2c^2)(E_+-E_-)}{Ep_0c}+\frac{(E_+^3-E_-^3)}{3Ep_0c}\right]\frac{dk}{k} \quad \ldots \quad (54)$$

For small velocity of the electron it reduces to

$$W_s = \frac{e^2}{\pi\hbar c}mc^2\left(1+\frac{p_0^2}{6m^2c^2}\right)\operatorname*{Lt}_{k\to\infty}\log\frac{k}{mc}+\text{finite terms} \ldots \quad \ldots \quad (55)$$

(*b*) Original Theory:

In the case of one electron theory we have on this model

$$W_s = \sum_k \frac{2\pi e^2\hbar^2 c}{k}\left\{\frac{\big(u(p_0)^*\alpha_k u(p_0-k)\big)\big(u(p_0-k)^*\alpha_k u(p_0)\big)-\big(u(p_0)^*u(p_0-k)\big)\big(u(p_0-k)^*u(p_0)\big)}{E(p_0)-ck-E(p_0-k)}\right.$$
$$\left.+\frac{\big(u(p_0)^*\alpha_k\underline{u}(p_0-k)\big)\big(\underline{u}(p_0-k)^*\alpha_k u(p_0)\big)-\big(u(p_0)^*\underline{u}(p_0-k)\big)\big(\underline{u}(p_0-k)^*u(p)\big)}{E(p_0)-ck+E(p_0-k)}\right\} (56)$$

$$= \frac{\pi e^2 \hbar^2}{ck^3} Sp \left\{ \frac{\big(E(p_0)+ck\big)\big(E(p_0)+H(p_0)\big)-H(p_0-k)\big(E(p_0)+H(p_0)\big)}{2E(p_0)} \right\} \quad (57)$$

Summing over the photons we finally obtain the standard result

$$W_s = \frac{e^2}{\pi\hbar}\int_0^\infty dk, \quad \ldots \quad (58)$$

which is highly divergent.

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