ON THE CLASS-NUMBER OF THE CORPUS $P(\sqrt{-k})$.

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§1. In a paper carrying the title of the present paper, Littlewood (1928) proved: If the extended Riemann hypothesis, (e.R.h.) is true, there exist infinitely many k such that

$$L(1) = \sum_{1}^{\infty} \frac{x(n)}{n} > \{1+0(1)\}\ e^{C} \log \log k$$

where $x(n)$ is a real primitive character (mod k).

This result was proved by Walfisz (1942) *without assuming the e.R.h.* His proof is based on the so-called 'class-number relations' discovered by Kronecker. In this paper I use the method developed by me in my paper 'An improvement of a theorem of Linnik and Walfisz' to give another proof of the result *without assuming the e.R.h.*

§2. Throughout we use the notation of my paper 3. In the definitions we only change the definition of b so that

$$\left(\frac{b}{p_r}\right) = +1 \text{ for } 1 \leqq r \leqq (g-1),$$

$$\left(\frac{b}{p_g}\right) = -1,$$

$$b \equiv 1 \pmod 8,$$

$$1 < b < 8a;$$

as before,

$$(1) \qquad T(x) = \sum_{x < n \leqq 2x} \ \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{8an+b}{m}\right);$$

also $S(x)$ is the same sum but with m going up to $x^{\frac{1}{4}}$ in the inner sum. The difference between $T(x)$ and $S(x)$ is of the order $x^{\frac{4}{5}}$, as proved in my paper 3.

The sum $S(x)$ is split up as before:

$$S(x) = S_1(x) + S_2(x) + S_3(x).$$

We find that

$$S_1(x) \sim \tfrac{1}{2} e^{C} x \ (\log \log x),$$

$$S_2(x) = O(x^{\frac{1}{4}}),$$

$$S_3(x) = O\left(\frac{x(\log \log x)^2}{\log x}\right);$$

the sums $S_2(x)$ and $S_3(x)$ are estimated in exactly the same way as in my paper 3, $S_1(x)$ only being different. Thus we have

(A) $$T(x) \sim \tfrac{1}{2}e^C x \log\log x.$$

We now write

(2) $$T(x) = T_1(x) + T_2(x)$$

where $T_1(x)$ is defined by (1) with the difference that in the outer sum n is restricted to take values such that $(8an+b)$ is quadratfrei; $T_2(x)$ is defined by the right-hand side of (1) but with n running through values in which $(8an+b)$ is divisible by a square greater than 1. We proceed to estimate $T_2(x)$. Now the numbers $(8an+b)$ cannot be divisible by p_r^2 unless $r > g$. The number of numbers $(8an+b)$ when $x < n \leqq 2x$ such that $8an+b$ is divisible by p_r^2 $(r > g)$ is clearly of the order

$$\sum_{r > g} \left(\frac{x}{p_r^2}\right) = O\left(\sum_{n > g} \frac{x}{n^2 \log^2 n}\right)$$

(3) $$= O\left(\frac{x}{g \log^2 g}\right) = O\left\{\frac{x(\log\log x)^2}{\log x(\log\log x)^2}\right\} = O\left(\frac{x}{\log x}\right).$$

Again, as observed by Davenport, we have

$$\sum_{n=u}^{v} x(n) = O(\sqrt{k} \log k)$$

where $x(n)$ is any non-principal character (mod k). It follows, since $a < x^{\frac{1}{30}}$ $(x > x_0)$ proved in my paper 3, that

(4) $$\sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{8an+b}{m}\right) = O(\log x)$$

for every n with $x < n \leqq 2x$. It now follows from (4) and (3) that

(5) $$T_2(x) = O\left(\frac{x}{\log x} \log x\right) = O(x).$$

From (A), (2) and (5) we finally get

(6) $$T_1(x) \sim \tfrac{1}{2}e^C x \log\log x,$$

i.e.

(7) $$\sum_{\substack{x < n \leqq 2x \\ (8an+b) \text{ quadratfrei}}} \sum_{m=1}^{\infty} \frac{1}{m} \cdot \left(\frac{8an+b}{m}\right) \sim \tfrac{1}{2}e^C x \log\log x.$$

Since 'almost all' $(8an+b)$ are quadratfrei when $x < n \leqq 2x$, it follows from (7) that there exists a positive integer n with $x < n \leqq 2x$ and such that $(8an+b)$ is quadratfrei and

(8) $$\sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{8an+b}{m}\right) > \tfrac{1}{2}e^C \{1 + 0(1)\} \log\log (8an+b)$$

since $\log \log x \sim \log \log (8an+b)$. From the Reciprocity Law for Jacobi's symbol

$$\left(\frac{8an+b}{m}\right) = \left(\frac{m}{8an+b}\right) \text{ when } m \equiv 1 \pmod 2,$$

$$\left(\frac{8an+b}{m}\right) = O \text{ when } m \equiv O \pmod 2,$$

by definition. Hence (8) becomes

$$\sum_{\text{mod } d} \frac{1}{m}\left(\frac{m}{8an+b}\right) > \frac{1}{2} e^{C} \{1+0(1)\} \log \log (8an+b)\,; \tag{9}$$

now

$$\left(\frac{2}{8an+b}\right) = +1 \text{ since } b \equiv 1 \pmod 8.$$

Hence

$$\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{m}{8an+b}\right) = \left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right) \sum_{\text{mod } d} \left(\frac{m}{8an+b}\right)\frac{1}{m}$$

$$= 2 \sum_{\text{mod } d} \frac{1}{m}\left(\frac{m}{8an+b}\right). \tag{10}$$

From (9) and (10) we get

$$\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{m}{8an+b}\right) > e^{C} \{1+0(1)\} \log \log (8an+b) \tag{11}$$

for suitable n with $x < n \leqq 2x$, and for all $x > x_0$ since $(8an+b)$ is quadratfrei in (8), (9), (10), (11) we finally get

Theorem 1. For all $x > x_0$ *there exists a quadratfrei number* $k = 8an+b$ *where* $x < n \leqq 2x$ *such that*

$$\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{m}{k}\right) > \{1+0(1)\}\, e^{C} \log \log k. \tag{12}$$

Since $\left(\frac{m}{k}\right)$ is a real primitive character (mod k) in (12), on account of k being quadratfrei, we can write Theorem 1 as

Theorem 2. For all $x > x_0$ *there exists a number* k *between* x *and* x^2 *and such that*

$$\sum_{1}^{\infty} \frac{x(n)}{n} > \{1+0(1)\}\, e^{C} \log \log k$$

where $x(n)$ *is a real primitive character* (mod k). Thus Littlewood's Theorem (and more) has been proved without assuming the *e.R.h.*

§3. References.

Littlewood, J. E. (1928). 'On the class-number of the corpus $P(\sqrt{-k})$'. *Proc. London Math. Soc.*, **27**, 358–372.

Walfisz, A. (1942). 'Über die Klassenzahl binarer quadratis chen Formen'. *Trav. Inst. Math. Tbilissi*, **11**, 57–71.

[Note: In a paper entitled 'On the k-analogue of a result in the theory of the Riemann Zeta function', *Mathematische Zeitschrift* (1934), Band **38**, 483–487, I have proved that

$$\sum_{1}^{\infty} \frac{x(n)}{n} = \Omega_R (\log \log k)$$

where $x(n)$ is a real primitive character (mod k).]

Note added during proof correction (May 7, 1947).

My paper 'An improvement of a theorem of Linnik and Walfisz' has been accepted for publication by the London Math. Society. There is *very little* difference between the arguments of the present papers and those of the paper to be published by the London Math. Society.