WIDTH OF NUCLEAR LEVELS.

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INTRODUCTION.

Bohr (1936) has emphasised that the problem of nuclear dynamics is essentially a many-body problem and that for a proper understanding of nuclear transmutations we must regard the process as happening in two distinct stages. On account of the close packing and intimate coupling of the nuclear particles the incident particle on colliding with the nucleus immediately shares its energy with the other constituent particles of the bombarded nucleus and gets amalgamated with them, the whole system forming what is called an intermediate compound nucleus. Each of the constituent particles in the compound nucleus will have some energy but not, in general, sufficient to enable it to escape from the rest. It is only when the energy gets *by chance* concentrated on any of them that it is enabled to escape from the rest. Or it might happen that before the energy gets concentrated on any one of the particles constituting the compound nucleus, the system gets down to a stable state either by the emission of radiation (radiative capture) or by breaking into two lighter nuclei (fission). Thus the result of subsequent breaking-up of the intermediate compound nucleus will depend upon a competition between the various disintegration—(including scattering), radiation- and fission-processes which are, of course, consistent with the conservation laws.

Just as in atomic theory the probability of an atom in an excited state emitting radiation depends upon the width of the level, the width of the intermediate compound nucleus gives us the probability of the emission of particles of any kind—neutrons, protons, alpha-particles, photons, etc. On this view the result of the competition between the various processes of disintegration, etc., of the compound nucleus will depend upon the relative partial widths of the level for the various processes. Thus it is that a knowledge of the position and the widths of the levels of the intermediate compound nucleus becomes very important for calculating the cross-section for any nuclear reaction.

The width of a nuclear level will depend (among other things) upon the energy of the level, its angular momentum and energy and angular momenta of the products of disintegration. To find this dependence we must introduce a special model of the nucleus and then solve its wave equation. If we could do this for the case of the intermediate compound nucleus, then at distances very great as compared with the nuclear radius some wave function should correspond to the splitting up of the compound nucleus, i.e. represent the products of disintegration. The calculation for the width of the level for any particular process of disintegration will then be nothing more than a mere evaluation of the matrix element corresponding to a transition from one state of the system to another. But unfortunately at the moment we not only do not know the Hamiltonian for a nucleus (all that we know is that the specifically nuclear force is a short range force of the exchange type; we do not know its dependence upon distance) but, even if we knew it, we know no method of solving a many-body problem when the coupling between the particles is very intimate.

Sometimes, however, it is required to know the dependence of the width primarily upon the energy and angular momentum of the expelled particle and it is the object of this paper to calculate a limiting value for the width of a nuclear level for very slow particles.

CALCULATION OF THE WIDTH.

It has already been pointed out that inside a nucleus a particle, be it a neutron or a proton, loses on account of the very close coupling its identity by sharing its energy with the other constituent particles. It is just for this reason that we cannot regard the two-body approximation as any good approximation at all. Nevertheless when the particle is outside, i.e. beyond the range of nuclear forces, which it would be if it is at a distance greater than r_0, the nuclear radius, the two-body approximation becomes a very good one and we can write at once for its equation (radial part only).

$$(1) \qquad \left[\frac{d^2}{dr^2}+k^2-V-\frac{l(l+1)}{r^2}\right]\phi^l(r) = 0.$$

Where all energies are expressed in units of $2M/\hbar^2$, M being the effective mass of the outgoing particle. The other symbols have their usual meaning. This equation does not hold for the region $r < r_0$, so that the usual boundary condition $\phi^l(0) = 0$ will have to be replaced by a suitable condition at $r = r_0$. If we knew the solution for the region $r < r_0$ this will be fairly straightforward, for all we will have to do will be to join smoothly at $r = r_0$ the solution for the region $r > r_0$ with that for the region $r < r_0$. Since even the equation for the interior of the nucleus is not known, we cannot follow this straight course, and so we will suppose that at the boundary $r = r_0$ the condition to be satisfied by the solution of (1) is

$$(2) \qquad \left[\frac{r}{\phi^l}\frac{d\phi^l}{dr}\right]_{r=r_0} = A.$$

The value of A will depend among other things on the energy of the particle. Our object is to find an expression for the width of the nuclear level in terms of this boundary condition or its derivative with respect to energy. The method we adopt to find the level-width is to calculate the cross-section at exact resonance and then vary the energy of the incident particle till the cross-section is reduced to one-half its value at resonance. The interval through which the energy of the particle has to be varied to reduce the cross-section to one-half its value at exact resonance gives us the half-width for that particular process. We will simplify matters further by considering the case of slow neutrons—incidently the most interesting case. Now the solution of (1) will be a linear combination of the regular and the irregular solutions, viz.

$$(3) \qquad \phi^l(r) = \left(\frac{\pi r}{2k}\right)^{1/2}\left[aJ_{l+1/2}(kr)+bJ_{-l-1/2}(kr)\right].$$

Where the coefficients a and b depend upon the energy of the escaping particle. If we consider the case of the protons, equation (1) will involve the coulomb potential as well, and consequently the solution will be in terms of the confluent hypergeometric series.

If $kr_0 \ll 1$, then in the vicinity of $r = r_0$ (3) can be written as

$$(4) \qquad \phi^l(r) = \left\{a(kr)^{l+1}+b(kr)^{-l}\right\}/k$$

from which we get

$$\left[\frac{r}{\phi^l}\frac{d\phi^l}{dr}\right]_{r=r_0} = A = \frac{(l+1)a(kr_0)^{l+1} - lb(kr_0)^{-l}}{a(kr_0)^{l+1} + b(kr_0)^{-l}}$$

or

$$b/a = (kr_0)^{l+1}(l+1-A)/(l+A). \tag{5}$$

If at very large distances we write the solution of (1) as

$$\phi^l(r) \sim \frac{j}{k}\sin\left(kr - \frac{l\pi}{2} + \delta_l\right) \tag{6}$$

we know that δ_l is connected with the coefficients a and b by the relation

$$\tan\delta_l = b/a \tag{7}$$

and that the contribution to the cross-section by particles having an orbital angular momentum l is

$$\sigma^l = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l.$$

This with the help of (7) becomes

$$\sigma^l = 4\pi(2l+1)b^2/k^2(a^2+b^2). \tag{8}$$

Evidently σ^l will be a maximum, i.e. we will get the case of resonance if $a = 0$, i.e. *if*

$$A_{\text{res}} = A_0 = -l. \tag{9}$$

Starting with the energy of the particle corresponding to resonance if we change the energy (of the particle) the coefficient of the regular solution in (3) namely a will begin to be different from zero till for a certain value of the energy of the particle a equals b. When this happens, the value of the cross-section as given by (8) reduces to one-half its value at exact resonance. In other words

$$b = a \tag{10}$$

is the condition for obtaining the half-width.

So long as the width of nuclear energy levels is small compared to the spacing between them we may write to a first approximation for the value of the boundary condition A in the immediate neighbourhood of resonance

$$A = A_0 + \Delta E\frac{dA}{dE} = -l + \Delta E\frac{dA}{dE}. \tag{11}$$

Substituting this in (5) we obtain

$$b/a = (kr_0)^{2l+1}\frac{(2l+1) - \Delta E\frac{dA}{dE}}{\Delta E\, dA/dE}. \tag{12}$$

The condition (10) now enables us to find the value of ΔE through which the relative energy of the particle must vary for the cross-section to become one-half its value at exact resonance i.e. the half-width.

We thus obtain

$$\Gamma^l = \Delta E = \frac{(2l+1)(kr_0)^{2l+1}}{\{1+(kr_0)^{2l+1}\}dA/dE}. \tag{13}$$

CALCULATION OF dA/dE; RESIDUAL NUCLEUS LEFT IN THE GROUND STATE.

To find the value of dA/dE in the immediate neighbourhood of resonance let us consider the wave equation (2) for a slightly different value of the energy k', say, of the escaping particle and denote the wave function for this case by $\phi'^{l}(r)$. Multiplying the equation for ϕ^l by ϕ'^l and the equation for ϕ'^l by ϕ^l and subtracting we obtain

$$\frac{d}{dr}\left[\phi^l \frac{d\phi'^l}{dr} - \phi'^l \frac{d\phi}{dr}\right] = -(k'^2-k^2)\phi^l\phi'^l.$$

Integrating and then multiplying by $r/\phi^l\phi'^l$ we obtain

$$\frac{r}{E-E'}\left[\frac{1}{\phi'^l}\frac{d\phi'^l}{dr} - \frac{1}{\phi^l}\frac{d\phi^l}{dr}\right] = -\frac{2M}{\hbar^2}\frac{r}{\phi^l\phi'^l}\int \phi^l\phi'^l\,dr. \tag{14}$$

Proceeding to the limit when $k' \to k$ we obtain at $r = r_0$

$$\frac{d}{dE}\left(\frac{r}{\phi^l}\frac{d\phi^l}{dr}\right)_{r=r_0} = -\frac{2M}{\hbar^2}r_0^2\{\textstyle\int(\phi^l)^2 dr/r_0(\phi^l)^2\}. \tag{15}$$

Taking the factor $\frac{\int(\phi^l)^2 dr}{r_0(\phi^l)^2}$ in (15), which in general will be less than unity, to be unity as an approximation we are left with

$$\Gamma^l = \frac{(2l+1)(kr_0)^{2l+1}}{\{1+(kr_0)^{2l+1}\}}\cdot\frac{\hbar^2}{2Mr_0^2}. \tag{16}$$

CALCULATION OF dA/dE; RESIDUAL NUCLEUS LEFT IN A NUMBER OF EXCITED STATES.

In deducing (15) we have not taken into account the possibility of the residual nucleus being left in a number of excited states. Let us now take this possibility into account and see how the value of dA/dE is affected. If $\boldsymbol{x}$ (x_1, x_2, x_3) stands for the co-ordinates of the escaping particle and y for all the parameters that may be necessary to describe the rest of the nucleus, the Schrodinger equation for the system is

$$\left[\frac{\hbar^2}{2M}\Delta_x^2 - H_y + E - V(r, y)\right]\Psi(\boldsymbol{x}, y) = 0, \tag{17}$$

where H_y is the Hamiltonian for the rest of the nucleus and $r = |\boldsymbol{x}|$. Let us write

$$\Psi(\boldsymbol{x}, y) = \sum_{i,l}\frac{1}{r}\phi_i^l(r)Y_l(\theta, \phi)\psi_i(y), \tag{18}$$

where ϕ_i's are the solutions of the radial part of the wave equation for the escaping particle in the two-body approximation, viz.:

$$\left[\frac{d^2}{dr^2} + E_p - V(r, y)\right]\phi_i^l = 0. \tag{19}$$

The centrifugal potential term is absorbed in $V(r, y)$ and the ψ_i's are the solutions of the wave equation for the residual nucleus, viz.:

$$(H_y - E_i)\psi_i = 0. \tag{20}$$

If we now multiply (17) by the angle function and integrate over the entire angle space for the escaping particle we get

$$(21)\quad \left\{\frac{\hbar^2}{2M}\frac{d^2}{dr^2}+(E-E_j)-V(r,y)\right\} w_j(r,y) + \sum_{i\neq j}\left\{\frac{\hbar^2}{2M}\frac{d^2}{dr^2}+(E-E_i)-V(r,y)\right\} w_i(r,y) = 0$$

where

$$(22)\qquad w_i(r,y) = \phi_i^l(r)\psi_i(y)$$

and the centrifugal potential term has been absorbed in $V(r, y)$.

For a slightly different value of the energy of the escaping particle, the nucleus being left in the same energy state, we would get

$$(23)\quad \left\{\frac{\hbar^2}{2M}\frac{d^2}{dr^2}+(E'-E_j)-V(r,y)\right\} w_j'(r,y) + \sum_{i\neq j}\left\{\frac{\hbar^2}{2M}\frac{d^2}{dr^2}+(E'-E_i)-V(r,y)\right\} w_i'(r,y) = 0.$$

Multiplying (21) by w'_j and (23) by w_j and subtracting we get after making use of (22) and (19).

$$(24)\qquad \sum_i \frac{w_j}{w_i}\left\{\frac{\hbar^2}{2M}\frac{d}{dr}\left(w_i'\frac{dw_i}{dr}-w_i\frac{dw_i'}{dr}\right)+(E-E')w_iw_i'\right\} = 0.$$

Writing $E-E' = E_p-E'_p = \Delta E_p$ where E_p stands for the particle energy, we get on integration with respect to r

$$(25)\quad w_j w_j'\left(\frac{1}{w_j}\frac{dw_j}{dr}-\frac{1}{w_j'}\frac{dw_j'}{dr}\right)+\frac{2M}{\hbar^2}\sum_i \Delta E_p\int w_j w_i'\,dr + \sum_{i\neq j}\left\{w_j w_i'\left(\frac{1}{w_i}\frac{dw_i}{dr}-\frac{1}{w_i'}\frac{dw_i'}{dr}\right)-\int w_i w_i'\left(\frac{1}{w_i}\frac{dw_i}{dr}-\frac{1}{w_i'}\frac{dw_i'}{dr}\right)\frac{d}{dr}(w_j/w_i)dr\right\}=0.$$

Now proceeding to the limit when $\Delta E_p \to 0$, i.e. the primed and the unprimed states become identical, we obtain after a little simplification

$$(26)\quad -\phi_j^{l2}\frac{d}{dE_p}\left(\frac{1}{\phi_j^l}\frac{d\phi_j^l}{dr}\right) = \frac{2M}{\hbar^2}\int |\phi_j^l|^2\,dr + \sum_{i\neq j}\frac{\psi_i(y)}{\psi_j(y)}\left\{\frac{2M}{\hbar^2}\int \phi_i^l\phi_j^l\,dr + \phi_j^l\phi_i^l\frac{d}{dE_p}\left(\frac{1}{\phi_i^l}\frac{d\phi_i^l}{dr}\right)-\int |\phi_i^l|^2\frac{d}{dE_p}\left(\frac{1}{\phi_i^l}\frac{d\phi_i^l}{dr}\right)\frac{d}{dr}(\phi_j^l/\phi_i^l)dr\right\}.$$

Where there is only one state possible for the residual nucleus the sum $\sum_{i\neq j}$ does not give any contribution and we are left with an expression which is the same as (15).

Summary.

An expression for the neutron width of nuclear levels is obtained in terms of the kinetic energy and orbital angular momentum of the neutron. The method adopted is to calculate the cross-section at exact resonance and then vary the energy of the particle till the cross-section is reduced to one-half its value at exact resonance. This interval through which the energy of the particle has to be varied gives us the half-width of the level.

Reference.

Bohr, N. (1936). *Nature*, **137**, 344–348.