ON SOME NEW KERNELS AND FUNCTIONS SELF-RECIPROCAL IN THE HANKEL TRANSFORM.

By R. P. AGARWAL, *Lecturer in Mathematics, Lucknow University, Lucknow.*

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§1. Introduction.

Adopting the usual notation of Hardy and Titchmarsh a function $f(x)$ is said to be R_ν if it is self-reciprocal in the Hankel Transform of order ν, i.e.,

$$f(x) = x^{\frac{1}{2}} \int_0^\infty y^{\frac{1}{2}} J_\nu(xy)\, f(y)\, dy.$$

Starting with an R_ν function a number of theorems have been given from time to time for finding another R_ν function with the help of some kernel or other.

The object of this paper is to discover some new kernels by using Hardy's formula (1935)

$$g(x) = \int_0^\infty P_1(y)\, P_2(xy)\, dy$$

where $P_1(x)$ and $P_2(x)$ are two Fourier kernels.

The main results are stated in the form of three theorems. The importance of these theorems lies in the fact that they are useful in identifying the nature of the resultant of two kernels, as transforming a self-reciprocal function of a known order to another of a different order.

The theorems have been discussed in §2–§4. Only the proof of Theorem (1) is given in detail. The other two theorems can be proved similarly.

Finally, in §5–§7, I have used the kernels of §§2–4 to investigate certain new self-reciprocal functions.

§2. Theorem 1.

The resultant,

$$K(x) = \int_0^\infty P_1(y)\, P_2(xy)\, dy,$$

of two kernels of the form (Titchmarsh, 1937)

$$P_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{1}{4}+\frac{\nu}{2}+\frac{s}{2}\right) \Gamma\left(\frac{3}{4}+\frac{\mu}{2}-\frac{s}{2}\right) \chi(s) x^{-s}\, ds \quad \ldots \quad (2.1)$$

and

$$P_2(x) = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} 2^s \Gamma\left(\frac{1}{4}+\frac{\nu}{2}+\frac{s}{2}\right) \Gamma\left(\frac{1}{4}+\frac{\rho}{2}+\frac{s}{2}\right) \omega(s) x^{-s}\, ds \quad \ldots \quad (2.2)$$

where, $$0<c<1, \quad 0<c'<1$$

and $$\chi(s)=\chi(1-s), \quad \omega(s)=\omega(1-s)$$

is a kernel transforming R_μ to R_ρ and vice versa.

In order to prove this consider the two kernels (2.1) and (2.2).

The resultant is,

$$K(x)=\frac{1}{2\pi i}\int_0^\infty P_1(y)\,dy\int_{c'-i\infty}^{c'+i\infty}2^s\Gamma\left(\frac{1}{4}+\frac{\nu}{2}+\frac{s}{2}\right)\Gamma\left(\frac{1}{4}+\frac{\rho}{2}+\frac{s}{2}\right)\omega(s)x^{-s}y^{-s}\,ds$$

$$=\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}2^s\Gamma\left(\frac{1}{4}+\frac{\nu}{2}+\frac{s}{2}\right)\Gamma\left(\frac{1}{4}+\frac{\rho}{2}+\frac{s}{2}\right)\omega(s)x^{-s}\,ds\int_0^\infty P_1(y)y^{-s}\,dy \tag{2.3}$$

provided the change in the order of integration is permissible.

Now, applying Mellin's Inversion formula to (2.1) and changing s to $(1-s)$ in it, we get

$$\int_0^\infty P_1(y)y^{-s}\,dy=\Gamma\left(\frac{3}{4}+\frac{\nu}{2}-\frac{s}{2}\right)\Gamma\left(\frac{1}{4}+\frac{\mu}{2}+\frac{s}{2}\right)\chi(1-s).$$

Therefore, from (2.3)

$$K(x)=\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}2^s\Gamma\left(\frac{1}{4}+\frac{\nu}{2}+\frac{s}{2}\right)\Gamma\left(\frac{1}{4}+\frac{\rho}{2}+\frac{s}{2}\right)\Gamma\left(\frac{3}{4}+\frac{\nu}{2}-\frac{s}{2}\right)\Gamma\left(\frac{1}{4}+\frac{\mu}{2}+\frac{s}{2}\right)\omega(s)\times$$
$$\chi(1-s)x^{-s}\,ds$$

$$=\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}2^s\Gamma\left(\frac{1}{4}+\frac{\rho}{2}+\frac{s}{2}\right)\Gamma\left(\frac{1}{4}+\frac{\mu}{2}+\frac{s}{2}\right)\psi(s)x^{-s}\,ds \quad \ldots \quad \ldots \tag{2.4}$$

where

$$\psi(s)=\Gamma\left(\frac{1}{4}+\frac{\nu}{2}+\frac{s}{2}\right)\Gamma\left(\frac{3}{4}+\frac{\nu}{2}-\frac{s}{2}\right)\omega(s)\chi(1-s).$$

Since, $\psi(s)=\psi(1-s)$ we have that $K(x)$ is a kernel transforming R_μ to R_ρ and *vice versa.*

It only remains to justify the change in the order of integration in (2.3).

Putting $s=c+it$ and applying the formula,

$$\left|\Gamma\left(A+\frac{it}{2}\right)\right|\sim ce^{-\frac{\pi}{4}|t|}\,|t|^{A-\frac{1}{2}}$$

to the integral (2.3) we see that it is less than

$$\left(\frac{2}{x}\right)^c\int_{-\infty}^{\infty}e^{-\frac{\pi}{2}|t|}\,|t|^{Rl\left[\frac{\nu+\rho}{2}\right]-\frac{1}{2}+c}\,|\,\omega(c+it)\,|\,dt\int_0^\infty|\,P_1(y)\,|\,y^{-c}\,dy.$$

Since, $$\omega(c+it)=O(e^{[\pi/2-\alpha+\eta]|t|})$$

we have that the t-integral is uniformly convergent. Also, the y-integral converges uniformly since $P_1(y)$ belongs to the class of analytic functions $A(\omega, \alpha)$. Also since, (2.4) exists, the change in the order of integration is justified.

Example :—To illustrate the above theorem, let us take,

$$\chi(s) = \frac{1}{2\Gamma\left(1+\frac{\mu+\nu}{2}\right)}, \qquad 0 \leq -\frac{1}{2}-\mu < Rl(s) < \frac{1}{2}-\frac{\mu}{2}+\frac{\nu}{2}.$$

Then (2.1) gives

$$P_1(x) = \frac{x^{\mu+\frac{1}{2}}}{(1+x^2)^{\frac{\mu+\nu}{2}+1}},$$

a known kernel transforming R_μ to R_ν (Baily, 1931).
Again, putting

$$\omega(s) = \frac{1}{\Gamma\left(\frac{\mu}{2}-\frac{s}{2}+\frac{7}{4}\right)\Gamma\left(\frac{\mu}{2}+\frac{s}{2}+\frac{5}{4}\right)},$$

$$0 < Rl(s) + \mu + \frac{1}{2} < \mu + \frac{5}{2}$$

and $$\nu \equiv \mu+2$$

we have from (2.2),

$$P_2(x) = x^{-\frac{1}{2}} J_{\mu+1}(x),$$

a known kernel transforming R_μ to $R_{\mu+2}$ and *vice versa* (Varma, 1939).

The resultant kernel is given by,

$$K(x) = x^{-\frac{1}{2}} \int_0^\infty \frac{y^\mu J_{\mu+1}(xy)}{(1+y^2)^{\frac{\mu+\nu}{2}+1}}\, dy \text{ (Watson, 1944)}$$

$$= \frac{x^{\mu+\frac{1}{2}}\Gamma(\mu+1)\Gamma\left(\frac{\nu-\mu}{2}\right)}{2^{\mu+2}\Gamma\left(\frac{\mu+\nu}{2}+1\right)\Gamma(\mu+2)}\, {}_1F_2\left(\mu+1\,;\, \frac{\mu-\nu}{2}+1, \mu+2\,;\, \frac{x^2}{4}\right)$$

$$+ \frac{x^{\nu+\frac{1}{2}}\Gamma\left(\frac{\mu-\nu}{2}\right)}{2^{\nu+2}\Gamma\left(\frac{\mu+\nu}{2}+2\right)}\, {}_1F_2\left(\frac{\mu+\nu}{2}+1\,;\, \frac{\mu+\nu}{2}+2, \frac{\nu-\mu}{2}+1\,;\, \frac{x^2}{4}\right)$$

valid for $\nu \neq \mu\,;\ Rl(\mu) > -1\,;\ Rl(\nu) > -1,\ Rl\left(\nu+2\mu+\frac{9}{2}\right) > 0.$

Our theorem at once gives that $K(x)$ is a kernel transforming R_ν to $R_{\mu+2}$ and *vice versa*.

§3. Theorem 2.

The resultant,

$$K(x) = \int_0^\infty P_1(xy)\, P_2(y)\, dy$$

of two kernels of the form,

$$P_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^s \Gamma\left(\frac{1}{4}+\frac{\mu}{2}+\frac{s}{2}\right)\Gamma\left(\frac{1}{4}+\frac{\nu}{2}+\frac{s}{2}\right)\chi(s)x^{-s}\,ds \quad \ldots \quad (3.1)$$

and,

$$P_2(x) = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} 2^s \Gamma\left(\frac{1}{4}+\frac{\mu}{2}+\frac{s}{2}\right)\Gamma\left(\frac{1}{4}+\frac{\lambda}{2}+\frac{s}{2}\right)\omega(s)x^{-s}\,ds \quad \ldots \quad (3.2)$$

where,

$$0 < c < 1, \quad 0 < c' < 1\,;$$

$$\omega(s) = \omega(1-s), \quad \chi(s) = \chi(1-s)$$

is a kernel transforming R_ν to R_λ.

In order to illustrate the above theorem, let us take in (3.1),

$$\chi(s) = \frac{1}{4\sqrt{2}\,\Gamma\left(\frac{\mu}{2}+\frac{3}{4}-\frac{s}{2}\right)\left(\frac{\mu}{2}+\frac{1}{4}+\frac{s}{2}\right)\left(\frac{\mu+\nu}{2}+1\right)},$$

$$Rl\left(-\nu-\frac{3}{2}\right) < Rl(s) < Rl(\mu)+\frac{3}{2}$$

and replace μ by $(\mu+2)$; we get

$$P_1(x) = \frac{x^{\mu+\frac{1}{2}}\Gamma(\mu+1)\Gamma\left(\frac{\nu-\mu}{2}\right)}{2^{\mu+2}\Gamma\left(\frac{\mu+\nu}{2}+1\right)\Gamma(\mu+2)}\ {}_1F_2\left(\mu+1\,;\,\frac{\mu-\nu}{2}+1,\,\mu+2\,;\,\frac{x^2}{4}\right)$$

$$+\frac{x^{\nu+\frac{1}{2}}\Gamma\left(\frac{\mu-\nu}{2}\right)}{2^{\nu+2}\Gamma\left(\frac{\mu+\nu}{2}+2\right)}\ {}_1F_2\left(\frac{\mu+\nu}{2}+1\,;\,\frac{\mu+\nu}{2}+2,\,\frac{\nu-\mu}{2}+1\,;\,\frac{x^2}{4}\right)$$

a kernel transforming R_ν to $R_{\mu+2}$ and *vice versa*. Again taking,

$$\omega(s) = 2^{\frac{\mu+\nu}{2}-\frac{3}{2}} \text{ in (3.2)}, \; Rl(s)+\frac{1}{2} > Rl(-\mu)$$

we get

$$P_2(x) = x^{\frac{\mu+\nu}{2}+\frac{1}{2}}K_{\frac{1}{2}[\nu-\mu]}(x),$$

a known kernel transforming R_μ to R_ν and *vice versa*.

The resultant kernel is given by

$$K(x) = \int_0^\infty \left[\frac{(xy)^{\mu+\frac{1}{2}}\Gamma(\mu+1)\Gamma\left(\frac{\nu-\mu}{2}\right)}{2^{\mu+2}\Gamma(\mu+2)\Gamma\left(\frac{\mu+\nu}{2}+1\right)} {}_1F_2\left(\mu+1\,;\frac{\mu-\nu}{2}+1,\, \mu+2\,; \frac{x^2y^2}{4}\right) \right.$$

$$\left. + \frac{(xy)^{\nu+\frac{1}{2}}\Gamma\left(\frac{\mu-\nu}{2}\right)}{2^{\nu+2}\Gamma\left(\frac{\mu+\nu}{2}+2\right)} {}_1F_2\left(\frac{\mu+\nu}{2}+1\,; \frac{\mu+\nu}{2}+2,\, \frac{\nu-\mu}{2}+1\,; \frac{x^2y^2}{4}\right) \right] \times$$

$$y^{\frac{1}{2}(\mu+\nu+1)} K_{\frac{1}{2}(\nu-\mu)}(y)\, dy$$

Expanding the ${}_1F_2$ function and integrating term by term, by the help of the integral,

$$\int_0^\infty t^{\mu-1} K_\nu(t)\, dt = 2^{\mu-2}\Gamma\left(\frac{\mu-\nu}{2}\right)\Gamma\left(\frac{\mu+\nu}{2}\right), \qquad Rl(\mu) > |Rl(\nu)|.$$

we get the new kernel,

$$K(x) = \frac{\pi \operatorname{cosec} \pi\left(\frac{\nu-\mu}{2}\right)}{2^{2-\frac{\mu+\nu}{2}}\Gamma\left(\frac{\mu+\nu}{2}+1\right)} \left\{ x^{\mu+\frac{1}{2}} {}_3f_2{}^*\left(\begin{matrix} \mu+1,\, \mu+1,\, \frac{\mu+\nu}{2}+1\,; \\ \frac{\mu-\nu}{2}+1,\, \mu+2\,; \end{matrix}\; x^2 \right) \right.$$

$$\left. - x^{\nu+\frac{1}{2}}\, {}_3f_2\left(\begin{matrix} \frac{\mu+\nu}{2}+1,\, \frac{\mu+\nu}{2}+1,\, \nu+1\,; \\ \frac{\mu+\nu}{2}+2,\, \frac{\nu-\mu}{2}+1\,; \end{matrix}\; x^2 \right) \right\}$$

valid for,

$$Rl(\mu) > -1,\ Rl(\mu+\nu+2) > 0,\ |x| < 1,\ \nu \neq \mu.$$

Our theorem gives that $K(x)$ is a kernel transforming $R_{\mu+2}$ to R_μ.

The term by term integration is valid since :

(i) ${}_1F_2$ is an integral function and hence is uniformly convergent in any arbitrary interval (O, A) of x

(ii) $$K_\nu(x) = O(x^{-\frac{1}{2}}e^{-x}) \text{ for } |x| \text{ large,}$$

(iii) The integrated series of ${}_3F_2$ functions is uniformly convergent for $|x| < 1$.

§4. Theorem 3.

The resultant,

$$K(x) = \frac{1}{2\pi i}\int_0^\infty P_1(xy)\, P_2(y)\, dy,$$

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* I shall henceforth use the abbreviated notation ${}_pf_q(a_1 \ldots a_p\,;\ b_1 \ldots b_q\,;\ x)$ to denote

$$\frac{\Gamma(a_1)\ldots\Gamma(a_p)}{\Gamma(b_1)\ldots\Gamma(b_q)}\, {}_pF_q(a_1\ldots a_p\,;\ b_1\ldots b_q\,;\ x).$$

of two kernels of the form,

$$P_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{1}{4}+\frac{\nu'}{2}+\frac{s}{2}\right)\Gamma\left(\frac{3}{4}+\frac{\rho}{2}-\frac{s}{2}\right)\omega(s)x^{-s}\,ds \qquad (4.1)$$

and,

$$P_2(x) = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \Gamma\left(\frac{1}{4}+\frac{\nu'}{2}+\frac{s}{2}\right)\Gamma\left(\frac{3}{4}+\frac{\mu'}{2}-\frac{s}{2}\right)\chi(s)x^{-s}\,ds \qquad (4.2)$$

where,

$$0<c<1,\ 0<c'<1;$$

$$\omega(s) = \omega(1-s),\ \chi(s) = \chi(1-s)$$

is a kernel transforming $R_{\mu'}$ to R_ρ.

To illustrate the above theorem, let us put in (4.1) and (4.2)

$$\omega(s) = \chi(s) = \frac{1}{2\Gamma\left(1+\frac{\mu+\nu}{2}\right)}\ ;\ 0\leq -\tfrac{1}{2}-\mu<Rl(s)<\tfrac{1}{2}-\frac{\mu-\nu}{2},$$

and

$$\nu' \equiv \mu,\ \rho = \mu' \equiv \nu,$$

then we have,

$$P_1(x) = P_2(x) = \frac{x^{\mu+\frac{1}{2}}}{(1+x^2)^{\frac{\mu+\nu}{2}+1}},$$

a known kernel transforming R_μ to R_ν.

The resultant kernel is given by

$$K(x) = x^{\mu+\frac{1}{2}}\int_0^\infty \frac{y^{2\mu+1}\,dy}{[(1+y^2)(1+x^2y^2)]^{\frac{\mu+\nu}{2}+1}}$$

$$= \frac{x^{\mu+\frac{1}{2}}}{2\pi i\ \Gamma\left(\frac{\mu+\nu}{2}+1\right)}\int_0^\infty \frac{y^{2\mu+1}dy}{(1+y^2)^{\frac{\mu+\nu}{2}+1}}\int_{-i\infty}^{i\infty}\Gamma\left(\frac{\mu+\nu}{2}+1+s\right)\Gamma(-s)(x^2y^2)^s\,ds.$$

Changing the order of integration and integrating, we get

$$\frac{x^{\mu+\frac{1}{2}}}{4\pi i\left\{\Gamma\left(\frac{\mu+\nu}{2}+1\right)\right\}^2}\int_{-i\infty}^{i\infty}\Gamma\left(\frac{\mu+\nu}{2}+1+s\right)\Gamma(\mu+1+s)\Gamma(-s)\Gamma\left(\frac{\nu-\mu}{2}-s\right)x^{2s}\,ds \qquad (4.3)$$

$$= \frac{x^{\mu+\frac{1}{2}}\pi \operatorname{cosec}\pi\left(\frac{\nu-\mu}{2}\right)}{2\left\{\Gamma\left(\frac{\mu+\nu}{2}+1\right)\right\}^2}\left\{x^{\nu-\mu}{}_2f_1\left(\nu+1,\ \frac{\mu+\nu}{2}+1;\ \frac{\nu-\mu}{2}+1;\ x^2\right)\right.$$

$$\left.-{}_2f_1\left(\mu+1,\ \frac{\mu+\nu}{2}+1;\ 1-\frac{\nu-\mu}{2};\ x^2\right)\right\}$$

valid for $Rl(\nu)>-1$, $Rl(\nu-\mu+1)>0$, $\nu\neq\mu$.

Our theorem at once gives that $K(x)$ is a kernel transforming R_ν into itself.

The change in the order of integration is permitted since the y and the s-integrals converge uniformly for $Rl(\nu-\mu+1)>0$ and $|x|<1$ respectively and also since the integral (4.3) exists.

§5. Let us take the $R_{\rho+2m}$ function,

$$2^{\frac{m+1}{2}} x^{m+\rho-\frac{1}{2}} e^{-\frac{1}{4}x^2} \quad W_{\rho+\frac{m+1}{2},\, \frac{1}{2}m}(\tfrac{1}{2}x^2) \quad .. \quad \text{(Varma, 1938)} \quad (5.1)$$

ρ being an even integer or zero and $\rho+2m>-1$.

Putting $\rho = \nu-2m$, we get the R_ν function

$$2^{\frac{m+1}{2}} x^{\nu-m-\frac{1}{2}} e^{-\frac{1}{4}x^2} \quad W_{\nu-\frac{3}{2}m+\frac{1}{2},\, \frac{1}{2}m}(\tfrac{1}{2}x^2)$$

Applying the kernel of §2 to this R_ν function we get the $R_{\mu+2}$ function,

$$F(x) = \frac{\pi \operatorname{cosec} \pi\left(\frac{\nu-\mu}{2}\right)}{\Gamma\left(\frac{\mu+\nu}{2}+1\right)} \int_0^\infty \left\{ \frac{(xy)^{\mu+\frac{1}{2}}}{2^{\mu+2}} {}_1f_2\left(\begin{matrix} \mu+1\,; \\ \frac{\mu-\nu}{2}+1,\ \mu+2\,; \end{matrix} \ \frac{x^2y^2}{4} \right) \right.$$

$$\left. -\frac{(xy)^{\nu+\frac{1}{2}}}{2^{\nu+2}} {}_1f_2\left(\begin{matrix} \frac{\mu+\nu}{2}+1\,; \\ \frac{\mu+\nu}{2}+2,\ \frac{\nu-\mu}{2}+1\,; \end{matrix} \ \frac{x^2y^2}{4} \right) \right\} \times 2^{\frac{m+1}{2}} y^{\nu-m-\frac{1}{2}} e^{-\frac{1}{4}y^2} W_{\nu-\frac{3}{2}m+\frac{1}{2},\, \frac{1}{2}m}(\tfrac{1}{2}x^2)$$

$$= \frac{\pi \operatorname{cosec} \pi\left(\frac{\nu-\mu}{2}\right)}{\Gamma\left(\frac{\mu+\nu}{2}+1\right)} \left\{ \frac{x^{\mu+\frac{1}{2}}}{2^{\mu-\frac{m}{2}+\frac{3}{2}}} \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n} \Gamma(\mu+1+n)}{n\,!\,\Gamma\left(\frac{\mu-\nu}{2}+1+n\right)\Gamma(\mu+2+n)} \right.$$

$$\times \int_0^\infty y^{\mu+\nu+2n-m} e^{-\frac{1}{4}y^2} W_{\nu-\frac{3}{2}m+\frac{1}{2},\, \frac{1}{2}m}(\tfrac{1}{2}y^2)\, dy$$

$$- \frac{x^{\nu+\frac{1}{2}}}{2^{\nu-\frac{m}{2}+\frac{3}{2}}} \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n} \Gamma\left(\frac{\mu+\nu}{2}+1+n\right)}{\Gamma\left(\frac{\mu+\nu}{2}+2+n\right)\Gamma\left(\frac{\nu-\mu}{2}+1+n\right) n\,!}$$

$$\left. \times \int_0^\infty y^{2\nu+2n-m} e^{-\frac{1}{4}y^2} W_{\nu-\frac{3}{2}m+\frac{1}{2},\, \frac{1}{2}m}(\tfrac{1}{2}y^2)\, dy \right\}$$

Changing the variable and applying Goldstein's result

$$\int_0^\infty x^{l-1} e^{-\frac{1}{2}x} W_{k,\,m}(x)\, dx = \frac{\Gamma(l+m+\frac{1}{2})\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)}$$

for $Rl(l \pm m+\frac{1}{2})>0$, we get

$$F(x)=\frac{\pi\operatorname{cosec}\pi\left(\frac{\nu-\mu}{2}\right)}{4\Gamma\left(\frac{\mu+\nu}{2}+1\right)}\left\{\begin{array}{l}\frac{x^{\mu+\frac{1}{2}}}{2^{\frac{\mu-\nu}{2}}}\,{}_3f_3\left(\begin{array}{l}\mu+1,\ \frac{\mu+\nu}{2}+1,\ \frac{\mu+\nu}{2}-m+1;\\ \frac{\mu-\nu}{2}+1,\ \mu+2,\ \frac{\mu-\nu}{2}+m+1;\end{array}\frac{x^2}{2}\right)\\ -\frac{x^{\nu+\frac{1}{2}}}{1}\,{}_3f_3\left(\begin{array}{l}\frac{\mu+\nu}{2}+1,\ \nu+1,\ \nu+1-m;\\ \frac{\mu+\nu}{2}+2,\ \frac{\nu-\mu}{2}+1,\ m+1;\end{array}\frac{x^2}{2}\right)\end{array}\right\} \quad \ldots (5.2)$$

where
$$Rl\left(\frac{\mu+\nu}{2}+1\right)>0,\quad Rl\left(\frac{\mu+\nu}{2}-m+1\right)>0,$$
$$Rl(\nu+1)>0,\quad Rl(\nu+1-m)>0$$

and $(\nu-2m)$ is an even integer or zero, and $2m \neq$ an integer or zero.

The term by term integration is valid, since,

(i) ${}_1F_2\,(a;\ b,\ c;\ x)$ is an integral function and hence is uniformly convergent in any arbitrary interval (O, A) of x.

(ii) $W_{k,m}\,(x) \sim O(x^k e^{-\frac{1}{2}x})$, for $|x|$ large.

(iii) The integrated series of ${}_3f_3$ functions is uniformly convergent.

Alternatively, applying the same kernel to the $R_{\mu+2}$ function (5.1), we get the R_ν function,

$$F(x)=\frac{\pi\operatorname{cosec}\pi\left(\frac{\nu-\mu}{2}\right)}{\Gamma\left(\frac{\mu+\nu}{2}+1\right)}\int_0^\infty\left\{\frac{(xy)^{\mu+\frac{1}{2}}}{2^{\mu-\frac{m}{2}+\frac{3}{2}}}\,{}_1f_2\left(\begin{array}{l}\mu+1;\\ \frac{\mu-\nu}{2}+1,\ \mu+2;\end{array}\frac{x^2y^2}{4}\right)-\frac{(xy)^{\nu+\frac{1}{2}}}{2^{\nu-\frac{m}{2}+\frac{3}{2}}}\,{}_1f_2\left(\begin{array}{l}\frac{\mu+\nu}{2}+1;\\ \frac{\mu+\nu}{2}+2,\ \frac{\nu-\mu}{2}+1;\end{array}\frac{x^2y^2}{4}\right)\right\}$$
$$y^{\mu+\frac{3}{2}-m}e^{-\frac{1}{4}y^2}W_{\mu-\frac{3}{2}m+\frac{5}{2},\frac{1}{2}m}(\tfrac{1}{2}y^2)\,dy$$

Integrating as before, we get,

$$F(x)=\frac{\pi\operatorname{cosec}\pi\left(\frac{\nu-\mu}{2}\right)}{2\Gamma\left(\frac{\mu+\nu}{2}+1\right)}\left\{\begin{array}{l}x^{\mu+\frac{1}{2}}\,{}_2f_2\left(\begin{array}{l}\mu+1,\ \mu+2-m;\\ \frac{\mu-\nu}{2}+1,\ m;\end{array}\frac{x^2}{2}\right)\\ -\frac{x^{\nu+\frac{1}{2}}}{2^{\frac{\nu-\mu}{2}}}\,{}_2f_2\left(\begin{array}{l}\frac{\mu+\nu}{2}+1,\ \frac{\mu+\nu}{2}+2-m;\\ 1+\frac{\nu-\mu}{2},\ \frac{\nu-\mu}{2}+m;\end{array}\frac{x^2}{2}\right)\end{array}\right\} \quad \ldots (5.3)$$

valid for,
$$Rl(\mu+2)>0,\quad Rl(\mu+2-m)>0$$
$$Rl\left(\frac{\mu+\nu}{2}+2\right)>0,\quad Rl\left(\frac{\mu+\nu}{2}+2-m\right)>0$$

and $(\mu+2-2m)$ is an even integer or zero and $\nu \neq \mu$.

The process of term by term integration is valid as before.

§ 6. Applying the kernel of § 3 to the $R_{\mu+2}$ function (Baily, 1930),

$$x^{\mu+\frac{5}{2}}e^{-\frac{1}{2}x^2}{}_1F_1\left(-2n\,;\,\frac{\mu}{2}+2-n\,;\,\tfrac{1}{2}x^2\right),$$

where n is a positive integer, we get the R_μ function

$$F(y)=\frac{\pi\operatorname{cosec}\pi\left(\frac{\nu-\mu}{2}\right)}{2^{2-\frac{\mu+\nu}{2}}\Gamma\left(\frac{\mu+\nu}{2}+1\right)}\int_0^\infty\left\{\begin{array}{l} x^{\mu+\frac{1}{2}}{}_3f_2\left(\begin{matrix}\mu+1,\,\mu+1,\frac{\mu+\nu}{2}+1\,; \\ \frac{\mu-\nu}{2}+1,\,\mu+2\,;\end{matrix}\;x^2\right) \\ -x^{\nu+\frac{1}{2}}{}_3f_2\left(\begin{matrix}\frac{\mu+\nu}{2}+1,\,\frac{\mu+\nu}{2}+1,\,\nu+1\,; \\ \frac{\mu+\nu}{2}+2,\,\frac{\nu-\mu}{2}+1\,;\end{matrix}\;x^2\right)\end{array}\right\}$$

$$\times(xy)^{\mu+\frac{5}{2}}e^{-\frac{1}{2}x^2y^2}{}_1F_1\left(-2n\,;\,\frac{\mu}{2}+2-n\,;\,\frac{1}{2}x^2y^2\right)dx$$

$$=\frac{\pi\operatorname{cosec}\pi\left(\frac{\nu-\mu}{2}\right)}{2^{2-\frac{\mu+\nu}{2}}\Gamma\left(\frac{\mu+\nu}{2}+1\right)}\left[\frac{y^{\mu+\frac{5}{2}}}{2\pi i}\int_0^\infty x^{2\mu+3}e^{-\frac{1}{2}x^2y^2}{}_1F_1\left(-2n;\;2+\frac{\mu}{2}-n;\;\tfrac{1}{2}x^2y^2\right)dx\right.$$

$$\times\int_{-i\infty}^{i\infty}\frac{\Gamma\left(\frac{\mu+\nu}{2}+1+s\right)\{\Gamma(\mu+1+s)\}^2\Gamma(-s)}{\Gamma(\mu+2+s)\Gamma\left(\frac{\mu-\nu}{2}+1+s\right)}(-x^2)^s\,ds$$

$$-\frac{y^{\mu+\frac{5}{2}}}{2\pi i}\int_0^\infty x^{\mu+\nu+3}e^{-\frac{1}{2}x^2y^2}{}_1F_1\left(-2n\,;\;2+\frac{\mu}{2}-n;\;\tfrac{1}{2}x^2y^2\right)dx$$

$$\left.\times\int_{-i\infty}^{i\infty}\frac{\left\{\Gamma\left(\frac{\mu+\nu}{2}+1+s\right)\right\}^2\Gamma(\nu+1+s)\Gamma(-s)(-x^2)^s}{\Gamma\left(\frac{\mu+\nu}{2}+2+s\right)\Gamma\left(\frac{\nu-\mu}{2}+1+s\right)}\,ds\right]$$

Changing the order of integration and integrating term by term we get

$$F(y)=\frac{\pi\operatorname{cosec}\pi\left(\frac{\nu-\mu}{2}\right)}{2^{2-\frac{\mu+\nu}{2}}\Gamma\left(\frac{\mu+\nu}{2}+1\right)}y^{\mu+\frac{5}{2}}\left[\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\{\Gamma(\mu+1+s)\}^2\Gamma\left(\frac{\mu+\nu}{2}+1+s\right)\Gamma(-s)(-)^s}{\Gamma(\mu+2+s)\Gamma\left(\frac{\mu-\nu}{2}+1+s\right)}ds\right.$$

$$\times\sum_{m=0}^{\infty}\frac{\Gamma(-2n+m)(y^2/2)^m}{m\,!\,\Gamma\left(2+\frac{\mu}{2}-n+m\right)}\int_0^\infty x^{2s+2m+2\mu+3}e^{-\frac{1}{2}x^2y^2}\,dx$$

$$-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\left\{\Gamma\left(\frac{\mu+\nu}{2}+1+s\right)\right\}^2\Gamma(\nu+1+s)\Gamma(-s)(-)^s}{\Gamma\left(\frac{\mu+\nu}{2}+2+s\right)\Gamma\left(\frac{\nu-\mu}{2}+1+s\right)}ds$$

$$\left.\times\sum_{m=0}^{\infty}\frac{\Gamma(-2n+m)(y^2/2)^m}{m!\;\Gamma\left(2+\frac{\mu}{2}-n+m\right)}\int_0^{\infty}x^{2s+2m+\mu+\nu+3}\,e^{-\frac{1}{2}x^2y^2}\,dx\right]$$

$$=\frac{\pi\operatorname{cosec}\pi\left(\frac{\nu-\mu}{2}\right)}{2^{2-\frac{\mu+\nu}{2}}\,\Gamma\left(\frac{\mu+\nu}{2}+1\right)}$$

$$\times\left\{\frac{2^{\mu+1}y^{-\mu-\frac{3}{2}}}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\{\Gamma(\mu+1+s\}^2\Gamma\left(\frac{\mu+\nu}{2}+1+s\right)\Gamma(-s)(-)^s\left(\frac{2}{y^2}\right)^s}{\Gamma\left(\frac{\mu-\nu}{2}+1+s\right)\Gamma(\mu+2+s)}ds\right.$$

$$\times{}_2F_1\left(-2n,\;\mu+2+s\,;\;2+\frac{\mu}{2}-n\,;\;1\right)$$

$$-\frac{2^{\frac{\mu+\nu}{2}+1}y^{-\nu-\frac{3}{2}}}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\left\{\Gamma\left(\frac{\mu+\nu}{2}+1+s\right)\right\}^2\Gamma(\nu+1+s)\Gamma(-s)(-)^s\left(\frac{2}{y^2}\right)^s}{\Gamma\left(\frac{\mu+\nu}{2}+2+s\right)\Gamma\left(\frac{\nu-\mu}{2}+1+s\right)}ds$$

$$\left.\times{}_2F_1\left(-2n,\;\frac{\mu+\nu}{2}+2+s\,;\;2+\frac{\mu}{2}-n\,;\;1\right)\right\}\quad\ldots\quad(6.1)$$

$$=\frac{\pi\operatorname{cosec}\pi\left(\frac{\nu-\mu}{2}\right)\Gamma\left(2+\frac{\mu}{2}-n\right)}{2^{1-\frac{\mu+\nu}{2}}\,\Gamma\left(\frac{\mu+\nu}{2}+1\right)\Gamma\left(2+\frac{\mu}{2}+n\right)}$$

$$\times\left\{2^{\mu}y^{-\mu-\frac{3}{2}}{}_4f_3\left(\begin{matrix}\mu+1,\;\mu+1,\;\frac{\mu+\nu}{2}+1,\;1+\frac{\mu}{2}+n\,;\\ \frac{\mu-\nu}{2}+1,\;\mu+2,\;1+\frac{\mu}{2}-n\,;\end{matrix}\;\frac{2}{y^2}\right)\right.$$

$$\left.-2^{\frac{\mu+\nu}{2}}y^{-\nu-\frac{3}{2}}{}_4f_3\left(\begin{matrix}\frac{\mu+\nu}{2}+1,\;\frac{\mu+\nu}{2}+1,\;\nu+1,\;1+\frac{\nu}{2}+n\,;\\ \frac{\mu+\nu}{2}+2,\;\frac{\nu-\mu}{2}+1,\;1+\frac{\nu}{2}-n\,;\end{matrix}\;\frac{2}{y^2}\right)\right\}\quad\ldots\quad(6.2)$$

The integral being convergent for

$$Rl(\mu)>-\tfrac{1}{2},\qquad Rl(\mu+\nu+4)>0,$$

$$Rl(\nu)>Rl(\mu)\,;\; n>Rl(\nu),\text{ and }|y|>\sqrt{2}.$$

The change in the order of integration is valid, since,

(i) $$\int_0^\infty \left| x^{2\mu+3+2s}\, e^{-\frac{1}{4}x^2y^2}\, {}_1F_1\!\left(-2n\,;\ 2+\frac{\mu}{2}-n\,;\ \tfrac{1}{2}x^2y^2\right)\right| dx$$

is convergent for $Rl(\mu) > -2$ and $\frac{3}{2}\, Rl(\mu)+1-n < 0$.

(ii) $$\int_{-i\infty}^{i\infty} \left| \frac{\{\Gamma(\mu+1+s)\}^2\,\Gamma\!\left(\frac{\mu+\nu}{2}+1+s\right)\Gamma(-s)}{\Gamma\!\left(\frac{\mu-\nu}{2}+1+s\right)\Gamma(\mu+2+s)}\,(-x^2)^s \right| ds$$

is uniformly convergent for $|x| < 1$.

(iii) The repeated integrals exist.

Similar reasoning holds for the other integral.

The term by term integration is valid, since,

(i) ${}_1F_1$ is an integral function and hence is uniformly convergent in any arbitrary interval (O, A) of x.

(ii) The integrated function reduces to ${}_2F_1\,(a, b\,;\ c,\ 1)$.

§7. Lastly, applying the kernel of §4, to the known R_ν function

$$x^{\frac{1}{6}-\frac{\nu}{3}}\, e^{-\frac{1}{4}x^2}\, I_{\frac{\nu}{3}-\frac{1}{6}}\,(\tfrac{1}{4}x^2),$$

we get the R_ν function,

$$F(x) = \frac{\pi \operatorname{cosec} \pi\left(\frac{\nu-\mu}{2}\right)}{2\left\{\Gamma\!\left(\frac{\mu+\nu}{2}+1\right)\right\}^2} \int_0^\infty \left\{ \begin{array}{l} y^{\nu+\frac{1}{2}}\,{}_2f_1\!\left(\nu+1,\ \frac{\mu+\nu}{2}+1\,;\ \frac{\nu-\mu}{2}+1\,;\ y^2\right) \\ -y^{\mu+\frac{1}{2}}\,{}_2f_1\!\left(\mu+1,\ \frac{\mu+\nu}{2}+1\,;\ 1-\frac{\nu-\mu}{2}\,;\ y^2\right) \end{array} \right\}$$
$$\times (xy)^{\frac{1}{6}-\frac{\nu}{3}}\, e^{-\frac{1}{4}x^2y^2}\, I_{\frac{\nu}{3}-\frac{1}{6}}\,(\tfrac{1}{4}x^2y^2)\, dy \qquad .. \quad (7.1)$$

Using the formula,

$$\Gamma(\nu+1)\, e^{-z}\, I_\nu(z) = (z/2)^\nu\, {}_1F_1\,(\nu+\tfrac{1}{2}\,;\ 2\nu+1\,;\ -2z)$$

and substituting the integral (4.3) for the ${}_2f_1$ functions in (7.1), we get

$$F(x) = \frac{x^{\frac{1}{6}-\frac{\nu}{3}}}{4\pi i\left\{\Gamma\!\left(\frac{\mu+\nu}{2}+1\right)\right\}^2} \int_0^\infty y^{\mu-\frac{\nu}{3}+\frac{2}{3}} \left\{ \frac{(x^2y^2/8)^{\frac{\nu}{3}-\frac{1}{6}}}{\Gamma\!\left(\frac{\nu}{3}+\frac{5}{6}\right)} \times {}_1F_1\!\left(\frac{\nu}{3}+\frac{1}{3}\,;\ \frac{2\nu}{3}+\frac{2}{3}\,;\ -\frac{1}{2}x^2y^2\right) \right\} dy$$
$$\times \int_{-i\infty}^{i\infty} \Gamma\!\left(\frac{\mu+\nu}{2}+1+s\right)\Gamma(\mu+1+s)\Gamma(-s)\Gamma\!\left(\frac{\nu-\mu}{2}-s\right) y^{2s}\, ds.$$

$$F(x) = \frac{x^{\frac{\nu}{3}-\frac{1}{6}}}{2^{\nu+\frac{3}{2}}\left\{\Gamma\left(\frac{\mu+\nu}{2}+1\right)\right\}^2\Gamma\left(\frac{\nu}{3}+\frac{5}{6}\right)}\int_0^\infty y^{\mu+\frac{\nu}{3}+\frac{1}{3}}\,{}_1F_1\left(\frac{\nu}{3}+\frac{1}{3};\ \frac{2\nu}{3}+\frac{2}{3};\ -\frac{1}{2}x^2y^2\right)dy$$

$$\times\int_{-i\infty}^{i\infty}\Gamma\left(\frac{\mu+\nu}{2}+1+s\right)\Gamma(\mu+1+s)\Gamma(-s)\Gamma\left(\frac{\nu-\mu}{2}-s\right)y^{2s}\,ds \quad .. \quad (7.2)$$

Changing the order of integration, we get

$$F(x) = \frac{x^{\frac{\nu}{3}-\frac{1}{6}}}{2^{\nu+\frac{3}{2}}\left\{\Gamma\left(\frac{\mu+\nu}{2}+1\right)\right\}^2\Gamma\left(\frac{\nu}{3}+\frac{5}{6}\right)}$$

$$\times\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma\left(\frac{\mu+\nu}{2}+1+s\right)\Gamma(\mu+1+s)\Gamma(-s)\Gamma\left(\frac{\nu-\mu}{2}-s\right)ds$$

$$\times\int^{\infty} y^{\frac{\nu}{3}+\frac{1}{3}+\mu+2s}\,{}_1F_1\left(\frac{\nu}{3}+\frac{1}{3};\ \frac{2\nu}{3}+\frac{2}{3};\ -\frac{1}{2}x^2y^2\right)dy.$$

Since,

$${}_1F_1(\alpha\,;\ \beta\,;\ x) = \frac{\Gamma(\beta)e^x}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 t^{\beta-\alpha-1}(1-t)^{\alpha-1}e^{-xt}dt.$$

$$[Rl(\beta)>Rl(\alpha)>0]$$

we get, on substitution,

$$F(x) = \frac{x^{\frac{\nu}{3}-\frac{1}{6}}\,\Gamma\left(\frac{2\nu}{3}+\frac{2}{3}\right)}{2^{\nu+\frac{3}{2}}\left\{\Gamma\left(\frac{\mu+\nu}{2}+1\right)\right\}^2\Gamma\left(\frac{\nu}{3}+\frac{5}{6}\right)\left\{\Gamma\left(\frac{\nu}{3}+\frac{1}{3}\right)\right\}^2}$$

$$\times\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma\left(\frac{\mu+\nu}{2}+1+s\right)\Gamma(\mu+1+s)\Gamma(-s)\Gamma\left(\frac{\nu-\mu}{2}-s\right)ds$$

$$\times\int_0^\infty y^{\frac{\nu}{3}+\frac{1}{3}+\mu+2s}e^{-\frac{1}{2}x^2y^2}\,dy\int_0^1[t(1-t)]^{\frac{\nu}{3}-\frac{2}{3}}\,e^{\frac{1}{2}x^2y^2t}\,dt.$$

Again, changing the order in integration of the y and t integrals since both are uniformly and absolutely convergent, and integrating the y-integral, we get,

$$F(x) = \frac{x^{\frac{\nu}{3}-\frac{1}{6}}\,\Gamma\left(\frac{2\nu}{3}+\frac{2}{3}\right)2^{\frac{\mu}{2}-\frac{5\nu}{6}-\frac{5}{6}}}{\left[\Gamma\left(\frac{\mu+\nu}{2}+1\right)\Gamma\left(\frac{\nu}{3}+\frac{1}{3}\right)\right]^2\Gamma\left(\frac{\nu}{3}+\frac{5}{6}\right)}$$

$$\times \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} 2^s \Gamma\left(\frac{\mu+\nu}{2}+1+s\right)\Gamma(\mu+1+s)\Gamma(-s)\Gamma\left(\frac{\nu-\mu}{2}-s\right)ds$$

$$\times \int_0^1 \Gamma\left(\frac{\nu}{6}+\frac{\mu}{2}+s+\frac{2}{3}\right) x^{-\left(\frac{\nu}{3}+\mu+2s+\frac{4}{3}\right)} t^{\frac{\nu}{3}-\frac{2}{3}} (1-t)^{\frac{\nu}{6}-\frac{\mu}{2}-\frac{4}{3}-s}\, dt.$$

$$= \frac{x^{-\mu-\frac{3}{2}}\, 2^{\frac{\mu}{2}-\frac{\nu}{6}-\frac{7}{6}}}{\left\{\Gamma\left(\frac{\mu+\nu}{2}+1\right)\right\}^2}\cdot\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma\left(\frac{\mu+\nu}{2}+1+s\right)\Gamma(\mu+1+s)\Gamma\left(\frac{\nu}{6}+\frac{\mu}{2}+s+\frac{2}{3}\right)\Gamma(-s)$$

$$\times\Gamma\left(\frac{\nu}{6}-\frac{\mu}{2}-\frac{1}{3}-s\right)\left(\frac{2}{x^2}\right)^s ds.$$

Changing s to $(-s)$ and evaluating the residues at poles of the integrand which lie on the positive half of the real axis, we get

$$F(x) = \frac{2^{\frac{\mu}{2}-\frac{\nu}{6}-\frac{7}{6}}\, x^{-\mu-\frac{3}{2}}}{\left\{\Gamma\left(\frac{\mu+\nu}{2}+1\right)\right\}^2}$$

$$\times\left[\left(\frac{x^2}{2}\right)^{\frac{\mu+\nu}{2}+1}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\mu-\nu}{2}-n\right)\Gamma\left(\frac{\mu+\nu}{2}+1+n\right)\Gamma\left(-\frac{\nu}{3}-\frac{1}{3}-n\right)\Gamma\left(n+\frac{2\nu}{3}+\frac{2}{3}\right)}{n\,!}\left(-\frac{x^2}{2}\right)^n\right.$$

$$+\left(\frac{x^2}{2}\right)^{\mu+1}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\nu-\mu}{2}-n\right)\Gamma(\mu+1+n)\Gamma\left(\frac{\nu}{6}-\frac{\mu}{2}-\frac{1}{3}-n\right)\Gamma\left(\frac{\mu}{2}+\frac{\nu}{6}+\frac{2}{3}+n\right)}{n\,!}\left(-\frac{x^2}{2}\right)^n$$

$$\left.+\left(\frac{x^2}{2}\right)^{\frac{\mu}{2}+\frac{\nu}{6}+\frac{2}{3}}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\nu}{3}+\frac{1}{3}-n\right)\Gamma\left(\frac{\mu}{2}-\frac{\nu}{6}+\frac{1}{3}-n\right)\Gamma\left(\frac{\nu}{6}+\frac{2}{3}+\frac{\mu}{2}+n\right)\Gamma\left(\frac{\nu}{3}+\frac{1}{3}+n\right)}{n\,!}\left(-\frac{x^2}{2}\right)^n\right]$$

$$F(x) = \frac{2^{\frac{\mu}{2}-\frac{\nu}{6}-\frac{7}{6}}\, x^{-\mu-\frac{3}{2}}}{\left\{\Gamma\left(\frac{\mu+\nu}{2}+1\right)\right\}^2}$$

$$\times\pi^2\left[\left(\frac{x^2}{2}\right)^{\frac{\mu+\nu}{2}+1}\operatorname{cosec}\pi\left(\frac{\mu-\nu}{2}\right)\operatorname{cosec}\pi\left(-\frac{\nu}{3}-\frac{1}{3}\right){}_2f_2\left(\begin{matrix}\frac{\mu+\nu}{2}+1,\ \frac{2\nu}{2}+\frac{2}{3}\,;\\ \frac{\nu}{3}+\frac{4}{3},\ 1-\frac{\mu-\nu}{2}\,;\end{matrix}\ -\frac{x^2}{2}\right)\right.$$

$$+\left(\frac{x^2}{2}\right)^{\mu+1}\operatorname{cosec}\pi\left(\frac{\nu-\mu}{2}\right)\operatorname{cosec}\pi\left(\frac{\nu}{6}-\frac{\mu}{2}-\frac{1}{3}\right){}_2f_2\left(\begin{matrix}\mu+1,\ \frac{\mu}{2}+\frac{\nu}{6}+\frac{2}{3}\,;\\ 1-\frac{\nu-\mu}{2},\ \frac{4}{3}-\frac{\nu}{6}+\frac{\mu}{2}\,;\end{matrix}\ -\frac{x^2}{2}\right)$$

$$+\left(\frac{x^2}{2}\right)^{\frac{\nu}{6}+\frac{2}{3}+\frac{\mu}{2}} \operatorname{cosec}\pi\left(\frac{\nu}{3}+\frac{1}{3}\right)\operatorname{cosec}\pi\left(\frac{\mu}{2}-\frac{\nu}{6}+\frac{1}{3}\right){}_2J_2\left(\begin{matrix}\frac{\nu}{3}+\frac{1}{3}, \frac{\nu}{6}+\frac{2}{3}+\frac{\mu}{2}; \\ \frac{2}{3}-\frac{\nu}{3}, \frac{2}{3}-\frac{\mu}{2}+\frac{\nu}{6};\end{matrix} -\frac{x^2}{2}\right)\Bigg]$$

which is the required R_ν function valid for $Rl(\nu)>-1$ and $Rl(\mu)>-1$ and $Rl(\nu+3\mu+4)>0$ and $Rl(\nu-3\mu+1)>0$.

The change in the order of integration in (7.2) is permissible, since,

(i) $$\int_{-i\infty}^{i\infty}\left|\Gamma\left(\frac{\mu+\nu}{2}+1+s\right)\Gamma(\mu+1+s)\Gamma(-s)\Gamma\left(\frac{\nu-\mu}{2}-s\right)y^{2s}\right|ds$$

exists for $0 \leq y \leq A$.

(ii) $$\int_0^\infty\left|y^{\frac{\nu}{3}+\frac{1}{3}+\mu+2s}{}_1F_1\left(\frac{\nu}{3}+\frac{1}{3}; \frac{2\nu}{3}+\frac{2}{3}; -\frac{1}{2}x^2y^2\right)\right|dy$$

also exists for $Rl(3\mu+\nu+4)>0$ and $Rl(\nu-3\mu+1)>0$.

(iii) The integral (7.2) exists for $Rl(\nu)>-1$ and $Rl(3\mu+\nu+4)>0$.

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