ON REAL CONTINUOUS SOLUTIONS OF ALGEBRAIC DIFFERENCE EQUATIONS (II).

By S. M. Shah, *Muslim University, Aligarh.*

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1. Introduction.

Let $y(x)$ be a real continuous solution of an algebraic difference equation of the first order

$$(1) \qquad P(y(x+1), y(x), x) = \Sigma\, ax^{\alpha}\, y(x)^{\beta_0}\, y(x+1)^{\beta_1} = 0.$$

I have shown that* $\quad \liminf_{x\to\infty} \log\log |\, y(x)\,| / x < \infty.$

This relation is the best possible. If we suppose that

$$(2) \qquad y(x) \text{ is a monotone function of } x \text{ for } x > x_0,$$

then $\quad \limsup_{x\to\infty} \log\log |\, y(x)\,| / x < \infty.$

In this paper I replace the condition (2) by a weaker condition. We shall suppose, in what follows, that $y(x)$ is real and continuous† for $x > x_0$.

Theorem 1. *If $y(x)$, a solution of an algebraic difference equation $P = 0$ of the first order, be such that* ‡

$$(3) \qquad y(x+h) \geqslant y(x)/e_2(Bx), \qquad 0 \leqslant h \leqslant 1,\ x > x_0,$$

where B is a positive constant, then

$$y(x) < e_2(Ax) \text{ for all } x > x_0(A),$$

where A $(>B)$ *is a constant depending on the given equation.*

Remark. Solutions satisfying condition (3) may exist. Consider, for instance, the equation §

$$(4) \qquad \big(y(x)-x^b\big)\big(y(x+1)-(x+1)^b\big) = 0.$$

This equation is satisfied by a continuous function $y(x)$ defined as follows:

$$y(x) = x^b, \qquad \text{for } 2n-1 \leqslant x \leqslant 2n,$$

$$= (2n)^b + 2\left\{ \left(2n+\frac{1}{2}\right)^b e_2\left(2n+\frac{1}{2}\right) - (2n)^b \right\}(x-2n), \qquad \text{for } 2n \leqslant x \leqslant 2n+\frac{1}{2},$$

$$= (2n+1)^b + 2\left\{ (2n+1)^b - \left(2n+\frac{1}{2}\right)^b e_2\left(2n+\frac{1}{2}\right) \right\}(x-2n-1),$$

$$\text{for } 2n+\frac{1}{2} \leqslant x \leqslant 2n+1.$$

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* See Ref. 2, pp. 548-9. There is a misprint in the last line of p. 558. Read (29) instead of (30).

† x_0 is not necessarily the same at each occurrence.

‡ $e_2(x)$ denotes exp (exp x).

§ Ref. 2, p. 549.

This function $y(x)$ satisfies the condition

$$y(x+h) \geqslant y(x)/e_2(x), \qquad 0 < h \leqslant 1,\ x > x_0.$$

2. Equations with Polynomial Coefficients.

Let $P_r(x)$ and $Q(x)$ be polynomials in x of degree p_r and q respectively.

Theorem 2. *Let $\Psi(x)$ be any increasing function such that*

$$\lim_{x\to\infty} \Psi(x) = \infty, \quad \lim_{x\to\infty} \log \Psi(x)/\log x = 0$$

and let $y(x)$ be a solution of the equation

$$P_m(x)y(x+m) + \ldots . + P_0(x)y(x) = Q(x). \tag{5}$$

Let $$a = \max\{\,|p_0-p_m|, \ldots . |p_{m-1}-p_m|, |q-p_m|\,\}.$$

If $y(x)$ satisfies the condition

$$y(x+h) \geqslant y(x)/\Psi(x), \qquad \text{for } 0 \leqslant h \leqslant 1,\ x > x_0, \tag{6}$$

then $$y(x) < \exp(Ax \log x), \qquad \text{for } x > x_0(A),$$

where A is any constant greater than a.

Cor. *If $y(x)$, a non-decreasing function of x, be a solution of* (5), *then*

$$y(x) < \exp(Ax \log x), \qquad x > x_0(A),$$

where A is any constant greater than a.

Theorem 3. *If $y(x)$, a solution of the equation*

$$P_1(x)y(x+1) + P_0(x)y(x) = Q(x), \tag{7}$$

satisfies either the condition (6) *or the condition*

$$\liminf_{x\to\infty} y(x)\Psi(x) > 0,$$

then

$$y(x) < \exp(Ax \log x), \qquad \text{for } x > x_0(A), \tag{8}$$

where A is any constant greater than $\max\{\,|p_0-p_1|, |q-p_1|\,\}$.

Cor. *Any solution $y(x)$ of* (7) *satisfies*

$$\liminf_{x\to\infty} \log|y(x)|/x \log x \leqslant A.$$

If in the equation (7), $Q(x) = 0$ and

$$P_0(x)/P_1(x) = -c \prod_1^{p_0} (x-\alpha_r) \Big/ \prod_1^{p_1} (x-\beta_r)$$

then *

$$y(x) = \tilde{\omega}(x)c^x \prod_1^{p_0} \Gamma(x-\alpha_r) \Big/ \prod_1^{p_1} \Gamma(x-\beta_r),$$

where $\tilde{\omega}(x)$ is any arbitrary periodic function. Since $\log \Gamma(x) \sim x \log x$, we see that (8) gives a ‘ best possible ’ upper bound.

* Ref. 1, p. 327.

3. Equations of order m.

THEOREM 4. *Let*

(9) $$P(y(x+m), \ldots . y(x), x) = \Sigma a x^{\alpha} y(x)^{\beta_0} \ldots . y(x+m)^{\beta_m} = 0$$

be an algebraic difference equation of order m and let

$$T' = a' x^{\alpha'} y(x)^{\beta'_0} \ldots . . y(x+m)^{\beta'_m}$$

be the principal term and let the terms of the equation be so related that when $\beta'_m = \beta_m$ *then either*

$$\beta'_{m-1} > \beta_{m-1} \text{ or } \beta'_{m-2} = \beta_{m-1} \text{ and } \beta'_i \geqslant \beta_i, \qquad (i = 0, 1, 2, \ldots m-2).$$

If $y(x)$ *be a solution such that it satisfies the conditions* (3) *and*

(10) $$\log y(x)/\log x \to \infty \text{ as } x \to \infty,$$

then $$y(x) < e_2(Ax), \text{ for all } x > x_0(A),$$

where $A(> B)$ *is a constant depending on the given equation.*

COR. 1. *If* $y(x)$, *a solution of an algebraic difference equation of the second order, satisfies the conditions* (3) *and* (10), *then* $y(x) < e_2(Ax)$, *for all* $x > x_0(A)$.

COR. 2. *If* $y(x)$, *a solution of an algebraic difference equation of the second order, satisfies the condition* (3), *then*

(11) $$y(x) < e_2(Ax), \text{ for a sequence of values of } x \to \infty.$$

4. Lemma 1.

If $f(x)$ *is continuous for* $x > x_0$ *and* $f(x) \geqslant \exp(Ax \log x)$, $A > 0$, *for a sequence of values of x tending to infinity, then there exists a sequence of numbers* x_n *tending to infinity such that*

$$f(x+1) \geqslant x^A f(x) \text{ for } x = x_1, x_2, \ldots \ldots$$

The proof is similar to that of Lemma 1 (Ref. 2, p. 550) and is omitted.

5. Lemma 2.

Let $f(x)$ *be continuous for* $x > x_0$ *and let*

(12) $$\limsup_{x \to \infty} \log \log f(x)/x \geqslant A > 0.$$

Let B be any positive constant less than A and suppose $f(x)$ *satisfies the condition*

(13) $$f(x+h) \geqslant f(x)/e_2(Bx), \qquad 0 \leqslant h \leqslant 1, x > x_0,$$

then there exists a sequence of numbers $x_n \to \infty$ *such that*

(14) $$f(x) \geqslant e_2(cx)\,;\ f(x+1) \geqslant \{f(x)\}^{\exp c},$$

where c is any positive constant such that $B < c < A$, *for* $x = x_1, x_2, \ldots \ldots$

PROOF. If $f(x) \geqslant e_2(cx)$ for all large x, then the result follows from Lemma 1 (Ref. 2, p. 550). Suppose therefore that $f(x) < e_2(cx)$ for a sequence of values of $x \to \infty$. Let $F(x) = f(x)/e_2(cx)$ and $X > x_0$ be an arbitrary large number such that $F(X) < 1$. Take $Y > X$ such that

$$\text{(i) } \log \log f(Y) > \tfrac{1}{2}(A+c)Y,$$

$$\text{(ii) } \frac{A-c}{2} Y + \log\left(1 - \frac{\exp(BY)}{\exp((A+c)Y/2)}\right) - c > 0.$$

This is possible since $\limsup_{x\to\infty} \log\log f(x)/x > (A+c)/2$. Hence $F(Y) > 1$. Let Z be the largest number such that $X < Z < Y$, and $F(Z) = 1$. Since $F(x)$ is continuous for $x \geqslant X$, Z exists and $F(x) \geqslant 1$ for $Z \leqslant x \leqslant Y$. Consider

$$f(Z),\ f(Z+1),\ \ldots\ldots f(Z+m)$$

where $Y \leqslant Z+m < Y+1$. Now

$$\begin{aligned}\log\log f(Z+m) &\geqslant \log\{\log f(Y)-\exp(BY)\}\\ &= \log\left\{\log f(Y)\left(1-\frac{\exp(BY)}{\log f(Y)}\right)\right\}\\ &= \log\log f(Y)+\log\left(1-\frac{\exp(BY)}{\log f(Y)}\right)\\ &> \frac{A+c}{2}Y+\log\left(1-\frac{\exp(BY)}{\exp((A+c)Y/2)}\right).\end{aligned}$$

Also

$$\begin{aligned}\frac{A+c}{2}Y &= cY+\frac{A-c}{2}Y > c(Z+m-1)+\frac{A-c}{2}Y\\ &= \log\log f(Z)+c(m-1)+\frac{A-c}{2}Y;\end{aligned}$$

and so from (ii)

$$\log\log f(Z+m)-\log\log f(Z) > cm.$$

Hence at least one of the values

$$\log\log f(Z+n)-\log\log f(Z+n-1), \qquad n = 1, 2, \ldots\ldots m,$$

is greater than c. If

$$\log\log f(Z+N)-\log\log f(Z+N-1) > c$$

then

$$f(Z+N-1) > e_2\big(c(Z+N-1)\big)$$
$$f(Z+N) \geqslant \{f(Z+N-1)\}^{\exp c}$$

which proves the lemma.

6. Lemma 3.

If $f(x)$ is continuous for $x > x_0$ and satisfies conditions (12) *and* (13), *then there exists a sequence of numbers $x_n \to \infty$ such that*

(15.1) $$f(x+1) \geqslant e_2(Dx)$$

(15.2) $$f(x+1) \geqslant \{f(x)\}^{\exp(D/2)}$$

(15.3) $$f(x+2) \geqslant \{f(x+1)\}^{\exp(D/2)}$$

for $x = x_1, x_2, \ldots\ldots$, D being any constant such that $B < D < A$.

The proof depends on the relation (14) proved above and is similar to that of Lemma 3 (Ref. 2, pp. 551-2).

REMARK. (i) If we assume that instead of condition (13) $f(x)$, in Lemma 2, satisfies

(13′) $$f(x+h) \geqslant f(x)/e_2(Ax), \qquad 0 \leqslant h \leqslant 1,\ x > x_0,$$

the conclusions (14), or (15), do not follow. Consider, for instance, $f(x)$ defined as follows. Let $x_n = n!^{n!}$ and

$$\begin{aligned} f(x) &= 1, \qquad \text{for } x_{n-1}+\tfrac{1}{4} \leqslant x \leqslant x_n-\tfrac{1}{4},\\ &= 1+4\big(e_2(x_n)-1\big)(x-x_n+\tfrac{1}{4}), \qquad \text{for } x_n-\tfrac{1}{4} \leqslant x \leqslant x_n,\\ &= 1-4\big(e_2(x_n)-1\big)(x-x_n-\tfrac{1}{4}), \qquad \text{for } x_n \leqslant x \leqslant x_n+\tfrac{1}{4}. \end{aligned}$$

Here $A = 1$ and (13′) is always satisfied. Further, (12) holds but there is no sequence $X_n \to \infty$ such that

$$f(x) \geqslant e_2(\delta x), f(x+1) \geqslant \{f(x)\}^{\exp\delta},$$

where δ is any positive number, for $x = X_1, X_2, \ldots\ldots$

(ii) Under the given hypotheses (12) and (13) it is not possible to obtain inequalities (14) satisfied for all $x > x_0$, or to obtain inequalities of the type (14), with c replaced by A, satisfied by a sequence of values of x tending to infinity. We can easily construct a suitable function by taking points on the curves $y = e_2(Ax)$, $y = e_2(Ax)/e_2(Bx)$ and joining them by a straight line.

7. LEMMA 4.

Let $\phi(x)$ be an increasing continuous function for $x > x_0$ such that

$$\lim_{x\to\infty} \phi(x) = \infty, \lim_{x\to\infty} \frac{\log \phi(x)}{x \log x} = c, \qquad 0 \leqslant c < \infty.$$

Let $f(x)$ be continuous and

$$f(x+h) \geqslant f(x)/\phi(x), \qquad 0 \leqslant h \leqslant 1,\ x > x_0,$$

and $\log f(x)/x \log x \geqslant A > c$, *for a sequence of values of $x \to \infty$, then there exists a sequence $x_n \to \infty$ such that*

$$f(x) \geqslant \exp(Bx \log x), \qquad f(x+1) \geqslant x^B f(x)$$

where B is any positive constant less than $A-c$, for $x = x_1, x_2, \ldots\ldots$

PROOF. Write $B = A-d$. Then $d > c$. If $f(x) \geqslant \exp(Bx \log x)$ for $x > x_0$, then the result follows from Lemma 1. Suppose therefore that $f(x) < \exp(Bx \log x)$ for a sequence of values of $x \to \infty$. Let $F(x) = f(x)/\exp(Bx \log x)$ and choose $X > x_0$ such that $F(X) < 1$. Take $Y > X$ such that

(i) $$\log \phi(Y) < \frac{d+c}{2} Y \log Y,$$

(ii) $$\frac{Y \log Y}{(Y+1) \log (Y+1)} > \frac{B}{\left(A - \frac{d+c}{2}\right)},$$

(iii) $$\log f(Y) \geqslant AY \log Y.$$

The result now follows as in Lemma 2.

8. Proof of Theorem 1.

Let T' denote the principal term and T any other term of $P=0$. Dividing throughout by T' we have ratios of the type

$$b\left\{\frac{x^{K_0}y(x)^{K_1}}{y(x+1)}\right\}^{\beta'_1-\beta_1},\qquad \beta'_1>\beta_1, \tag{16.1}$$

$$b\left\{\frac{x^{K_0}}{y(x)}\right\}^{\beta'_0-\beta_0},\qquad \beta'_0>\beta_0, \tag{16.2}$$

$$bx^{\alpha-\alpha'},\qquad \alpha'>\alpha. \tag{16.3}$$

Let K be the $\max|K_1|$ for all the ratios of type (16.1) and let $A=1+\max(1, B, \log K)$. If $y(x)\geqslant e_2(Ax)$ for a sequence of values of $x\to\infty$, then by Lemma 2, we can find a sequence $x_n\to\infty$ such that

$$y(x)\geqslant e_2(cx),\ y(x+1)\geqslant\{y(x)\}^{\exp c},$$

where $c=\frac{1}{2}+\max(1, B, \log K)$ for $x=x_n$. Hence the ratios of the types (16.1), (16.2) and also (16.3) tend to zero as $x=x_n\to\infty$, and so we have a contradiction. Hence $y(x)<e_2(Ax)$ for $x>x_0(A)$.

9. Proof of Theorem 2.

Let $A=k+a$ $(k>0)$. From the given equation we have

$$1+\frac{P_{m-1}(x)y(x+m-1)}{P_m(x)y(x+m)}+\ldots\ldots+\frac{P_0(x)y(x)}{P_m(x)y(x+m)}=\frac{Q(x)}{P_m(x)y(x+m)}. \tag{17}$$

If $y(x)\geqslant\exp(Ax\log x)$ for a sequence of values of $x=x_n\to\infty$, we have by Lemma 4

$$y(x+m)\geqslant(x+m-1)^By(x+m-1),$$

and

$$y(x+m-1)\geqslant\exp(x+m-1),$$

where $B=a+\frac{1}{2}k$, for a sequence of values of $x=X_n\to\infty$.
Hence if $x=X_n$, then

$$\left|\frac{P_{m-1}(x)y(x+m-2)}{P_m(x)y(x+m)}\right|<c_1x^{p_{m-1}-p_m}\frac{y(x+m-1)}{y(x+m)}$$

$$<c_1x^{p_{m-1}-p_m-B}\to 0,\quad \text{as } x=X_n\to\infty$$

$$\left|\frac{P_{m-2}(x)y(x+m-2)}{P_m(x)y(x+m)}\right|<c_1x^{p_{m-2}-p_m}\frac{y(x+m-1)\psi(x+m-2)}{y(x+m)}$$

$$<c_1x^{\left(p_{m-2}-p_m+\frac{k}{4}-B\right)}\to 0\quad \text{as } x=X_n\to\infty.$$

Similarly, the remaining ratios on the left-hand side of (17) tend to zero as $x=X_n\to\infty$. Further

$$\left|\frac{Q(x)}{P_m(x)y(x+m)}\right|<c_1x^{q-p_m}\frac{1}{y(x+m)}\to 0,\quad \text{as } x=X_n\to\infty.$$

Hence we have a contradiction and so $y(x)<\exp(Ax\log x)$, $x>x_0$ which proves the theorem.

The corollary follows from the theorem. Theorem 3 and its corollary follow immediately from Lemma 1.

10. Proof of Theorem 4.

The ratios T/T' take the forms

(18.1) $$b\left\{\frac{x^{K_0}y(x)^{K_1}\ldots\ldots y(x+m-1)^{K_m}}{y(x+m)}\right\}^{\beta'_m-\beta_m}, \quad \beta'_m > \beta_m,$$

(18.2) $$b\left\{\frac{x^{p_0}y(x)^{p_1}\ldots y(x+m-2)^{p_{m-1}}}{y(x+m-1)}\right\}^{\beta'_{m-1}-\beta_{m-1}}, \quad \beta'_{m-1} > \beta_{m-1},$$

(18.3) $$b\left\{\frac{x^{k_0}}{y(x)^{k_1}\ldots y(x+m-2)^{k_{m-1}}}\right\},$$

respectively, where $k_1, k_2, \ldots$ are all non-negative constants and k_0 is negative if all $k_1, \ldots k_{m-1}$ are zero. Let

$$|K_1| + \cdots + |K_m| = A_m,$$
$$|p_1| + \cdots + |p_{m-1}| = B_m;$$

and let α_1 be max A_m for all ratios of the type (18.1) and let α_2 be max B_m for all ratios of the type (18.2) and let $a = \max(1, B, \log\alpha_1, \log\alpha_2)$, $A = 1+2a$. If $y(x) \geqslant e_2(Ax)$ for a sequence of values of $x \to \infty$, then by Lemma 3 we can choose a sequence (x_n) such that

$$y(x+1) \geqslant e_2(Dx),$$
$$y(x+1) \geqslant \{y(x)\}^{\exp(D/2)},$$
$$y(x+2) \geqslant \{y(x+1)\}^{\exp(D/2)},$$

where $D = \frac{1}{2}+2a$, for $x = x_n \to \infty$. By hypothesis (10) all ratios of the type (18.3) tend to zero as $x \to \infty$. Further

$$y(x+m-2) \leqslant y(x+m-1)e_2(B(x+m-2)),$$
$$y(x+m-3) \leqslant y(x+m-2)e_2(B(x+m-3)),$$

and so on. Hence for $x = x_n+2-m$, we have

$$x^{K_0}y(x)^{K_1}\ldots y(x+m-1)^{K_m} < y(x+m-1)^{\alpha_1}\{e_2(B(x+m-1))\}^{\Delta},$$

where Δ is a constant. Further

$$x^{p_0}y(x)^{p_1}\ldots y(x+m-2)^{p_{m-1}} \leqslant y(x+m-2)^{\alpha_2}\{e_2(B(x+m-2))\}^{\Delta}.$$

Hence the ratios of the type (18.1) and (18.2) tend to zero as $x = x_n+2-m \to \infty$. Thus we have a contradiction and so $y(x) < e_2(Ax)$ for $x > x_0(A)$.

Corollary 1 follows immediately since when $\beta'_2 = \beta_2$ then either $\beta'_1 > \beta_1$ or $\beta'_1 = \beta_1$ and $\beta'_0 > \beta_0$.

To prove Corollary 2, suppose, if possible, that (11) is false. Then

(19) $$y(x) \geqslant e_2(Ax) \text{ for } x > x_0.$$

Hence both conditions (3) and (10) are satisfied and from Corollary 1 we have $y(x) < e_2(Ax)$ for $x > x_0$, which contradicts (19), and Corollary 2 is proved.

References.

1. Milne Thomson, L. M. (1933). Calculus of Finite Differences.
2. Shah, S. M. (1947). On Real Continuous Solutions of Algebraic Difference Equations. *Bull. Amer. Math. Soc.*, **53**, 548–558.