ON THE NÖRLUND SUMMABILITY OF DERIVED FOURIER SERIES.

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(*Received May 31; read August 5, 1949.*)

1.1. Let $f(x)$ be a function integrable in the sense of Lebesgue and periodic of period 2π. Let the Fourier series associated with $f(x)$ be

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx). \qquad (1)$$

The derived series of the Fourier series is

$$\sum_{n=1}^{\infty} n(b_n \cos nx-a_n \sin nx). \qquad (2)$$

It is well known that the derived series may not itself be a Fourier series.

The (C, r) summability of the series (2) corresponding to a function $f(x)$ of bounded variation has been considered by Young (1914) and M. Riesz (1923). They have proved:

A. The derived series corresponding to a function $f(x)$ of bounded variation is summable (C, r), $0 < r \leqq 1$ to $f'(x)$ at the point x for which

$$\chi_0(t) = \int_0^t |d_s\, \chi(s)| = o(t),$$

where $\chi(t) \equiv f(x+t)-f(x-t)-2t\, f'(x)$, i.e. for almost every x.

Hille and Tamarkin (1932) have extended the scope of this theorem by applying to the derived series the method of Nörlund summability which, as known, includes, as special cases, the method of Cesàro summability.

If we remove the restriction that $f(x)$ is of bounded variation in $(-\pi, \pi)$, then the theorem A as well as that of Hille and Tamarkin break down. The (C, r) summability of derived series when $f(x)$ is not necessarily a function of bounded variation, has been considered by Lebesgue (1905), Privaloff (1919), and Young (1914). They have proved the following theorem:

B. At every point x, where

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}$$

exists and is finite, the derived series (2) is summable (C, r), $r > 1$, to the value $f'(x)$.

The corresponding result for the case of Nörlund summability has been obtained by Astrachan (1936) who proved:

C. A regular Nörlund method of summation (N, p_n) sums the series (2) to $f'(x)$ at all the points at which

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}$$

exists and is finite, if the generating sequence (p_n) satisfies the following conditions :—

(i) $n^j|\Delta^{j-1}p_n| = O(P_n)$ $(j = 1, 2)$;

(ii) $\sum_{k=1}^{n} k^{j-m-1}(n-k)^m |\Delta^{j-1}p_k| = O(P_n)$ $(j = 1, 2, 3; m = 0, 1)$;

(iii) $\sum_{k=1}^{n} \frac{|P_k|}{k^2} = O\left(\frac{P_n}{n}\right)$.

The theorem B has been still more generalized by Chen (1929) who has established the following theorem :

D. At every point x, where

$$\chi_0(t) = \int_0^t u^{-1}|\chi(u)|\,du = o(t),$$

where $\chi(t) \equiv f(x+t)-f(x-t)-2t\,f'(x)$, the series (2) is summable (C, r), $r > 1$ to the value $f'(x)$.

The object of this paper is to extend still further the scope of summability of the derived Fourier series by applying the more general method of Nörlund summability. We prove the following:

THEOREM. *A regular Nörlund method of summation (N, p_n) sums the derived series,*

$$\sum_{n=1}^{\infty} n(b_n \cos nx - a_n \sin nx)$$

to $f'(x)$ at every point x at which

$$\chi_0(t) = \int_0^t u^{-1}|\chi(u)|\,du = o(t),$$

where $\chi(t) \equiv f(x+t)-f(x-t)-2tf'(x)$, if the generating sequence (p_n) satisfies the following conditions :—

(I) $\sum_{k=1}^{n} |\Delta p_k| = O\left(\frac{P_n}{n}\right)$;

(II) $\sum_{k=1}^{n} k|\Delta^2 p_k| = O\left(\frac{P_n}{n}\right)$;

(III) $\sum_{k=1}^{n} \frac{|P_k|}{k^2} = O\left(\frac{P_n}{n}\right)$.

It is evident that this theorem includes the previous theorems B, C and D.

1.2. In § 2.1 to § 2.3, we give the main proof of the theorem. It will be noted that this proof involves the help of a number of results whose proof, for the sake of convenience, will be given separately in the shape of lemmas.

2.1. PROOF. In what follows we shall suppose that the sequence (p_n) satisfies the conditions (I) to (III) of the above theorem.

The kth term of the series (2) is

$$\frac{1}{\pi}\int_0^{\pi} \{f(x+t)-f(x-t)\}\ k \sin kt\, dt,$$

and hence the nth generalized regular * Nörlund mean is

$$N'_n\left[f(x), p_n\right]=\frac{1}{\pi}\int_0^{\pi} \{f(x+t)-f(x-t)\}\cdot\frac{1}{P_n}\sum_{k=0}^{n} P_{n-k}\,(k \sin kt)\, dt$$

$$=\frac{1}{\pi}\int_0^{\pi} \chi(t)\cdot\frac{1}{P_n}\sum_{k=0}^{n} P_{n-k}\,(k \sin kt)\, dt$$

$$+\frac{2f'(x)}{\pi}\int_0^{\pi} t\cdot\frac{1}{P_n}\sum_{k=0}^{n} P_{n-k}\,(k \sin kt)\, dt.$$

Now

$$\int_0^{\pi} t\cdot\frac{1}{P_n}\sum_{k=0}^{n} P_{n-k}\,(k \sin kt)\, dt$$

$$=\left[\frac{t}{P_n}\sum_{k=0}^{n} P_{n-k}\,(-\cos kt)\right]_0^{\pi}+\int_0^{\pi}\frac{1}{P_n}\sum_{k=0}^{n} P_{n-k}\cos kt\, dt$$

$$=-\frac{\pi}{P_n}\sum_{k=0}^{n} P_{n-k}\,(-1)^k+\pi+\int_0^{\pi}\frac{1}{P_n}\sum_{k=0}^{n} P_{n-k}\cos kt\, dt$$

$$=-\frac{\pi}{P_n}\sum_{k=0}^{n} p_{n-k}\,\frac{1-(-1)^{k+1}}{2}+\pi+\left[\frac{1}{P_n}\sum_{k=1}^{n} P_{n-k}\frac{\sin kt}{k}\right]_0^{\pi}+o(1)$$

$$=\frac{\pi}{2}-\frac{\pi}{2}\,\frac{(-1)^n}{P_n}\sum_{k=0}^{n} p_k(-1)^k+o(1)$$

$$=\frac{\pi}{2}+o(1),$$

since

$$\sum_{k=0}^{n} p_k(-1)^k=o(P_n)\ \text{and}\ |P_n|\to\infty,$$

by virtue of lemmas 1 and 2.

Hence

$$N'_n\left[f(x), p_n\right]-f'(x)=\frac{1}{\pi}\int_0^{\pi}\chi(t)\cdot\frac{1}{P_n}\sum_{k=0}^{n} P_{n-k}(k \sin kt)\, dt+o(1).$$

* A Nörlund method of summation (N, p_n) is said to be *regular* if the following conditions are satisfied:

(i) $p_n=o(P_n)$, (ii) $\sum_{k=0}^{n} |p_k|=O(P_n)$.

In order to prove the theorem we have to show that the right-hand side integral is $o(1)$, as $n \to \infty$, under the conditions of the theorem.

Putting

$$N_n(t) = \frac{1}{\pi P_n} \sum_{k=0}^{n} P_{n-k} \cos kt,$$

we get

$$L = \frac{1}{\pi} \int_0^{\pi} \chi(t) \frac{1}{P_n} \sum_{k=0}^{n} P_{n-k}\, k \sin kt\, dt$$

$$= -\int_0^{\pi} \chi(t) N'_n(t)\, dt$$

$$= -\left(\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} + \int_{\delta}^{\pi} \right) \chi(t) \,.\, N'_n(t)\, dt$$

$$= -(L_1 + L_2 + L_3), \qquad \text{say.}$$

2.2. Now

$$L_1 = \int_0^{n^{-1}} t^{-1} \chi(t) \,.\, tN'_n(t)\, dt$$

$$= O\left(\int_0^{n^{-1}} t^{-1} |\chi(t)| \,.\, |tN'(t)|\, dt \right)$$

$$= O\left(n \int_0^{n^{-1}} t^{-1} |\chi(t)|\, dt \right)$$

$$= O\left(n\chi_0 (n^{-1}) \right)$$

$$= o(1), \qquad \text{as } n \to \infty,$$

since $tN'_n(t) = O(n)$, for all t such that $O \leqq t \leqq n^{-1}$.

It is known that (Astrachan, 1936)

$$tN'_n(t) = o(1)$$

uniformly in t for $O < \delta \leqq |t| \leqq \pi$, where δ is fixed, provided

(i) $|\Delta^{j-1} p_n| = o(P_n) \qquad (j = 1, 2),$

(ii) $\sum_{k=0}^{n} (n-k)^{\beta} |\Delta^2 p_k| = o(P_n) \qquad (\beta = 0, 1).$

Hence we have

$$L_3 = \int_{\delta}^{\pi} t^{-1}\chi(t) \,.\, tN'_n(t)\, dt$$

$$= o\left(\int_{\delta}^{\pi} | t^{-1}\chi(t) |\, dt \right)$$

$$= o(1), \text{ as } n \to \infty .$$

2.3. It remains to show that $L_2 = o(1)$, as $n \to \infty$. To prove this we require a suitable estimate for $tN'_n(t)$ in the interval (n^{-1}, δ). We make use of the following notations:

$$| p_n | = r_n, \quad R_n = r_0 + r_1 + \ldots\ldots + r_n ;$$

$$r(u) = r_{[u]},\ R(u) = R_{[u]},$$

where $[u]$ denotes the largest integer $\leqq u$;

$$W^{\beta}(m) = \sum_{k=0}^{m} (n-k)^{\beta} | \Delta^2 p_k | ; \qquad \tau = [t^{-1}].$$

We have (Astrachan, 1936)

$$| tN'_n(t) | \leqq A \sum_{m=1}^{4} M_{nm}(t) \qquad \text{for } n^{-1} \leqq t \leqq \delta,$$

where

$$M_{n1}(t) = \frac{1}{R(n)} \sum_{\beta=0}^{1} n^{\beta} t^{\beta-1} R(t^{-1}),$$

$$M_{n2}(t) = \frac{1}{R(n)} \sum_{j=1}^{2} \sum_{\beta=0}^{1} (n-\tau)^{\beta} t^{\beta-j-1} | \Delta^{j-1} p_{\tau} | ,$$

$$M_{n3}(t) = \frac{1}{R(n)} \sum_{j=1}^{2} t^{-j-1} | \Delta^{j-1} p_n | ,$$

$$M_{n4}(t) = \frac{1}{R(n)} \sum_{\beta=0}^{1} t^{\beta-3} \{ W^{\beta}(n) - W^{\beta}(t^{-1}) \} .$$

Now we have

$$L_2 = \int_{n^{-1}}^{\delta} t^{-1}\chi(t) \,.\, tN'_n(t)\, dt$$

$$= O\left(\int_{n^{-1}}^{\delta} t^{-1} | \chi(t) | \,.\, | tN'_n(t) |\, dt \right)$$

$$= O\left(\int_{n^{-1}}^{\delta} t^{-1}|\chi(t)| \cdot \sum_{m=1}^{4} M_{nm}(t)\,dt\right)$$

$$= O\left(\int_{n^{-1}}^{\delta} t^{-1}|\chi(t)|\,M_{n1}(t)\,dt\right)+O\left(\int_{n^{-1}}^{\delta} t^{-1}|\chi(t)|\,M_{n2}(t)\,dt\right)$$

$$+O\left(\int_{n^{-1}}^{\delta} t^{-1}|\chi(t)|\,M_{n3}(t)\,dt\right)+O\left(\int_{n^{-1}}^{\delta} t^{-1}|\chi(t)|\,M_{n4}(t)\,dt\right)$$

$$= O(L_{21})+O(L_{22})+O(L_{23})+O(L_{24}), \qquad \text{say.}$$

Now

$$L_{21} = \int_{n^{-1}}^{\delta} t^{-1}|\chi(t)| \cdot \frac{1}{R(n)} \sum_{\beta=0}^{1} n^{\beta} t^{-1+\beta} R(t^{-1})\,dt$$

$$= \int_{n^{-1}}^{\delta} t^{-1}|\chi(t)| \frac{R(t^{-1})}{tR(n)}\,dt + \int_{n^{-1}}^{\delta} t^{-1}|\chi(t)| \frac{nR(t^{-1})}{R(n)}\,dt$$

$$= \left[\chi_0(t)\frac{R(t^{-1})}{nR(n)}\right]_{n^{-1}}^{\delta} - \frac{1}{R(n)}\int_{n^{-1}}^{\delta} \chi_0(t)\frac{dR(t^{-1})}{t} + \frac{1}{R(n)}\int_{n^{-1}}^{\delta} \chi_0(t)R(t^{-1})\frac{dt}{t^2}$$

$$+\left[\chi_0(t)\frac{nR(t^{-1})}{R(n)}\right]_{n^{-1}}^{\delta} - \frac{n}{R(n)}\int_{n^{-1}}^{\delta} \chi_0(t)\,dR(t^{-1}$$

$$= O\left(\frac{1}{R(n)}\right)+O(n\chi_0(n^{-1}))+o\left(\frac{1}{R(n)}\int_{n^{-1}}^{\delta} |dR(t^{-1})|\right)+o\left(\frac{1}{R(n)}\int_{n^{-1}}^{\delta} R(t^{-1})\frac{dt}{t}\right)$$

$$+O\left(\frac{n}{R(n)}\right)+o\left(\frac{n}{R(n)}\int_{n^{-1}}^{\delta} t\,|dR(t^{-1})|\right)$$

$$= O\left(\frac{n}{R(n)}\right)+o\left(\frac{1}{R(n)}\int_{1}^{n} |dR(s)|\right)+o\left(\frac{1}{R(n)}\int_{1}^{n} \frac{R(s)}{s}\,ds\right)$$

$$+o\left(\frac{n}{R(n)}\int_{1}^{n} \frac{|dR(s)|}{s}\right)+o(1)$$

$$= O\left(\frac{n}{|P_n|}\right)+o\left(\frac{1}{|P_n|}\sum_{k=0}^{n}|p_k|\right)+o\left(\frac{1}{|P_n|}\sum_{k=1}^{n}\frac{|P_k|}{k}\right)+o\left(\frac{n}{|P_n|}\sum_{k=1}^{n}\frac{|p_k|}{k}\right)$$

$$+o(1).$$

$$= O\left(\frac{n}{|P_n|}\right)+o(1)+o\left(\frac{n}{|P_n|}\sum_{k=1}^{n}\frac{|P_k|}{k^2}\right)+o\left(\frac{n}{|P_n|}\sum_{k=1}^{n}\frac{|p_k|}{k}\right)+o(1),$$

the second term on the right being $o(1)$ by virtue of regularity condition.

Now we have

$$n = o(P_n) \quad \text{by virtue of lemma 2};$$

$$\sum_{k=1}^{n} \frac{|P_k|}{k^2} = O\left(\frac{P_n}{n}\right) \quad \text{by virtue of condition (III) of the theorem};$$

$$o\left(\frac{n}{|P_n|} \cdot \sum_{k=1}^{n} \frac{|p_k|}{k}\right) = o\left(\frac{n}{|P_n|} \sum_{k=1}^{n} \frac{|P_k|}{k^2}\right) = o(1) \quad \text{by virtue of lemma 3 and the}$$

condition (III) of the theorem.

Hence

$$L_{21} = o(1).$$

Again

$$L_{22} = \int_{n^{-1}}^{\delta} t^{-1} |\chi(t)| \cdot \frac{1}{R(n)} \sum_{j=1}^{2} \sum_{\beta=0}^{1} (n-\tau)^{\beta} t^{\beta-j-1} | \Delta^{j-1} p_{\tau} | \, dt$$

$$= \left[\chi_0(t) \cdot \frac{1}{R(n)} \sum_{j=1}^{2} \sum_{\beta=0}^{1} (n-\tau)^{\beta} t^{\beta-j-1} | \Delta^{j-1} p_{\tau} | \right]_{n^{-1}}^{\delta}$$

$$- \int_{n^{-1}}^{\delta} \chi_0(t) \cdot \frac{1}{R(n)} \sum_{j=1}^{2} \sum_{\beta=0}^{1} (\beta-j-1)(n-\tau)^{\beta} t^{\beta-j-2} | \Delta^{j-1} p_{\tau} | \, dt$$

$$- \int_{n^{-1}}^{\delta} \chi_0(t) \cdot \frac{1}{R(n)} \sum_{j=1}^{2} \sum_{\beta=0}^{1} (n-\tau)^{\beta} t^{\beta-j-1} d | \Delta^{j-1} p_{\tau} |$$

$$+ \int_{n^{-1}}^{\delta} \chi_0(t) \cdot \frac{1}{R(n)} \sum_{j=1}^{2} \sum_{\beta=0}^{1} \beta \, . \, t^{\beta-j-1} | \Delta^{j-1} p_{\tau} | \, d\tau$$

$$= O\left(\frac{1}{R(n)} \sum_{j=1}^{2} \sum_{\beta=0}^{1} n^{\beta}\right) + o\left(\frac{1}{R(n)} \sum_{j=1}^{2} n^{j} | \Delta^{j-1} p_n |\right)$$

$$+ o\left(\frac{1}{R(n)} \int_{n^{-1}}^{\delta} \sum_{j=1}^{2} \sum_{\beta=0}^{1} (n-\tau)^{\beta} t^{\beta-j-1} | \Delta^{j-1} p_{\tau} | \, dt\right)$$

$$+ o\left(\frac{1}{R(n)} \int_{n^{-1}}^{\delta} \sum_{j=1}^{2} \sum_{\beta=0}^{1} (n-\tau)^{\beta} t^{\beta-j} d | \Delta^{j-1} p_{\tau} |\right)$$

$$+ o\left(\frac{1}{R(n)} \int_{n^{-1}}^{\delta} \sum_{j=1}^{2} t^{1-j} | \Delta^{j-1} p_{\tau} | \, d\tau\right)$$

$$= O\left(\frac{n}{R(n)}\right) + o\left(\frac{1}{R(n)} \sum_{j=1}^{2} n^j | \Delta^{j-1} p_n |\right)$$

$$+ o\left(\frac{1}{R(n)} \int_1^n \sum_{j=1}^{2} \sum_{\beta=0}^{1} (n-[s])^\beta s^{j-\beta-1} | \Delta^{j-1} p_{[s]} | ds\right)$$

$$+ o\left(\frac{1}{R(n)} \int_1^n \sum_{j=1}^{2} \sum_{\beta=0}^{1} s^{j-\beta} (n-[s])^\beta d | \Delta^{j-1} p_{[s]} |\right)$$

$$+ o\left(\frac{1}{R(n)} \int_1^n \sum_{j=1}^{2} s^{j-1} | \Delta^{j-1} p_{[s]} | ds\right)$$

$$= O\left(\frac{n}{|P_n|}\right) + o\left(\frac{1}{|P_n|} \sum_{j=1}^{2} n^j | \Delta^{j-1} p_n |\right)$$

$$+ o\left(\sum_{j=1}^{2} \sum_{\beta=0}^{1} \frac{1}{|P_n|} \sum_{k=0} k^{j-\beta-1} (n-k)^\beta | \Delta^{j-1} p_k |\right)$$

$$+ o\left(\sum_{j=1}^{2} \sum_{\beta=0}^{1} \frac{1}{|P_n|} \sum_{k=1}^{n} k^{j-\beta} (n-k)^\beta | \Delta^j p_k |\right)$$

$$+ o\left(\sum_{j=1}^{2} \frac{1}{|P_n|} \sum_{k=1}^{n} k^{j-1} | \Delta^{j-1} p_k |\right)$$

$$= O\left(\frac{n}{|P_n|}\right) + o\left(\frac{n^2 | \Delta p_n |}{|P_n|}\right) + o\left(\frac{n | p_n |}{|P_n|}\right)$$

$$+ o\left(\sum_{j=0}^{2} \sum_{\beta=0}^{1} \frac{1}{|P_n|} \sum_{k=1}^{n} k^{j-\beta} (n-k)^\beta | \Delta^j p_k |\right).$$

Now we show that each term on the right in the above is $o(1)$. We have

$n = o(P_n)$, by virtue of lemma 2 ;

$n | p_n | = O(P_n)$ from lemma 3 ;

$n^2 | \Delta p_n | = O(P_n)$ from lemma 5.

We shall now show that

$$\sum_{k=1}^{n} k^{j-\beta} (n-k)^\beta | \Delta^j p_k | = O(P_n) \qquad (\beta = 0, 1 ; j = 0, 1, 2).$$

Breaking the summation on the left in six parts, we have

(a) $\sum_{k=1}^{n} | p_k | = O(P_n)$, (b) $\sum_{k=1}^{n} k | \Delta p_k | = O(P_n)$,

$$(c)\ \sum_{k=1}^{n} k^2 \mid \Delta^2 p_k \mid = O(P_n), \qquad (d)\ \sum_{k=1}^{n} k^{-1}(n-k) \mid p_k \mid = O(P_n),$$

$$(e)\ \sum_{k=1}^{n} (n-k) \mid \Delta p_k \mid = O(P_n), \qquad (f)\ \sum_{k=1}^{n} k(n-k) \mid \Delta^2 p_k \mid = O(P_n).$$

For, (*a*) follows from regularity condition, (*b*) and (*e*) from condition (I), and (*c*) and (*f*) from condition (II) of the theorem, and finally

$$\sum_{k=1}^{n} k^{-1}(n-k) \mid p_k \mid = \sum_{k=1}^{n} n \,.\, \frac{\mid p_k \mid}{k} - \sum_{k=1}^{n} \mid p_k \mid$$

$$\leqq C \sum_{k=1}^{n} \frac{n \mid P_k \mid}{k^2} - \sum_{k=1}^{n} \mid p_k \mid$$

$$= O(P_n),$$

by virtue of regularity condition, lemma 3, and condition (III) of the theorem.

Hence

$$L_{22} = o(1).$$

Similarly,

$$L_{23} = \int_{n^{-1}}^{\delta} t^{-1} \mid \chi(t) \mid \cdot \frac{1}{R(n)} \sum_{j=1}^{2} t^{-j-1} \mid \Delta^{j-1} p_n \mid dt$$

$$= \left[\chi_0(t)\ \frac{1}{R(n)} \sum_{j=1}^{2} t^{-j-1} \mid \Delta^{j-1} p_n \mid \right]_{n^{-1}}^{\delta}$$

$$+ (j+1) \int_{n^{-1}}^{\delta} \chi_0(t) \cdot \frac{1}{R(n)} \sum_{j=1}^{2} t^{-j-2} \mid \Delta^{j-1} p_n \mid dt$$

$$= O\left(\frac{1}{R(n)} \sum_{j=1}^{2} \mid \Delta^{j-1} p_n \mid \right) + o\left(\frac{1}{R(n)} \sum_{j=1}^{2} n^j \mid \Delta^{j-1} p_n \mid \right)$$

$$+ o\left(\int_{n^{-1}}^{\delta} \frac{1}{R(n)} \sum_{j=1}^{2} t^{-j-1} \mid \Delta^{j-1} p_n \mid dt \right)$$

$$= O\left(\frac{1}{R(n)} \sum_{j=1}^{2} \mid \Delta^{j-1} p_n \mid \right) + o\left(\frac{1}{R(n)} \sum_{j=1}^{2} n^j \mid \Delta^{j-1} p_n \mid \right)$$

$$+ o\left(\left[\frac{1}{R(n)} \sum_{j=1}^{2} \mid \Delta^{j-1} p_n \mid t^{-j} \right]_{n^{-1}}^{\delta} \right)$$

$$= O\left(\frac{1}{R(n)} \sum_{j=1}^{2} \mid \Delta^{j-1} p_n \mid \right) + o\left(\frac{1}{R(n)} \sum_{j=1}^{2} n^j \mid \Delta^{j-1} p_n \mid \right).$$

Now

$$|\Delta^{j-1}p_n| = o(P_n) \quad (j = 1, 2) \text{ by virtue of regularity condition};$$

$$n\,|p_n| = O(P_n) \quad \text{from lemma 3};$$

and $\quad n^2\,|\Delta p_n| = O(P_n)\quad$ follows from lemma 5.

Hence

$$L_{23} = o(1).$$

Finally,

$$L_{24} = \int_{n^{-1}}^{\delta} t^{-1}\,|\chi(t)|\cdot\frac{1}{R(n)}\sum_{\beta=0}^{1} t^{\beta-3}\{W^{\beta}(n)-W^{\beta}(t^{-1})\}\,dt$$

$$= \left[\chi_0(t)\,\frac{1}{R(n)}\sum_{\beta=0}^{1} t^{\beta-3}\{W^{\beta}(n)-W^{\beta}(t^{-1})\}\right]_{n^{-1}}^{\delta}$$

$$-(\beta-3)\int_{n^{-1}}^{\delta}\chi_0(t)\,\frac{1}{R(n)}\sum_{\beta=0}^{1} t^{\beta-4}\{W^{\beta}(n)-W^{\beta}(t^{-1})\}\,dt$$

$$+\int_{n^{-1}}^{\delta}\chi_0(t)\cdot\frac{1}{R(n)}\sum_{\beta=0}^{1} t^{\beta-3}\,dW^{\beta}(t^{-1})$$

$$= O\left(\frac{1}{R(n)}\sum_{\beta=0}^{1} W^{\beta}(n)\right)$$

$$+o\left(\int_{n^{-1}}^{\delta}\frac{1}{R(n)}\sum_{\beta=0}^{1} t^{\beta-3}\{W^{\beta}(n)-W^{\beta}(t^{-1})\}\,dt\right)$$

$$+o\left(\int_{n^{-1}}^{\delta}\frac{1}{R(n)}\sum_{\beta=0}^{1} t^{\beta-2}\,dW^{\beta}(t^{-1})\right)$$

$$= O\left(\frac{1}{R(n)}\sum_{\beta=0}^{1} W^{\beta}(n)\right)+o\left(\int_{1}^{n}\frac{1}{R(n)}\sum_{\beta=0}^{1} s^{-\beta+1}\{W^{\beta}(n)-W^{\beta}(s)\}\,ds\right)$$

$$+o\left(\int_{1}^{n}\frac{1}{R(n)}\sum_{\beta=0}^{1} s^{2-\beta}\,dW^{\beta}(s)\right).$$

Now

$$\int_{1}^{n}\frac{1}{R(n)}\sum_{\beta=0}^{1} s^{1-\beta}\{W^{\beta}(n)-W^{\beta}(s)\}\,ds = \left[\frac{1}{R(n)}\sum_{\beta=0}^{1}\frac{s^{2-\beta}}{2-\beta}\cdot\{W^{\beta}(n)-W^{\beta}(s)\}\right]_{1}^{n}$$

$$+\int_{1}^{n}\frac{1}{R(n)}\cdot\sum_{\beta=0}^{1}\frac{s^{2-\beta}}{2-\beta}\,dW^{\beta}(s)$$

$$= O\left(\frac{1}{R(n)} \sum_{\beta=0}^{1} W^{\beta}(n)\right)$$

$$+ O\left(\int_{1}^{n} \frac{1}{R(n)} \sum_{\beta=0}^{1} s^{2-\beta} dW^{\beta}(s)\right).$$

Hence

$$L_{24} = O\left(\frac{1}{R(n)} \sum_{\beta=0}^{1} W^{\beta}(n)\right) + o\left(\int_{1}^{n} \frac{1}{R(n)} \sum_{\beta=0}^{1} s^{2-\beta} dW^{\beta}(s)\right).$$

Now

$$\sum_{k=1}^{n} (n-k)^{\beta} \mid \Delta^2 p_k \mid = o(P_n) \qquad (\beta = 0,\ 1) \text{ by lemma 1 ;}$$

and

$$\sum_{k=1}^{n} k^{2-\beta}(n-k)^{\beta} \mid \Delta^2 p_k \mid = O(P_n) \qquad (\beta = 0,\ 1) \text{ follows from condition (II).}$$

Hence

$$L_{24} = o(1).$$

Combining the above results we obtain

$$L_2 = o(1), \quad \text{as } n \to \infty.$$

This completes the proof of the theorem.

3.1. LEMMAS. We now give the lemmas which have been utilised in establishing the theorem:

Lemma 1. If $\sum_{k=1}^{n} k \mid \Delta^2 p_k \mid = O\left(\frac{P_n}{n}\right)$ and $\sum_{k=1}^{n} \frac{\mid P_k \mid}{k^2} = O\left(\frac{P_n}{n}\right)$, then

$$\sum_{k=0}^{n} (n-k)^{\beta} \mid \Delta^2 p_k \mid = o(P_n) \quad \text{for } \beta = 0,\ 1.$$

Proof: The case for $\beta = 0$ follows directly from the condition

$$\sum_{k=1}^{n} k \mid \Delta^2 p_k \mid = O\left(\frac{P_n}{n}\right).$$

The case for $\beta = 1$ follows from a corresponding result of Astrachan (1936).

Lemma 2. Under conditions (II) and (III) of the theorem,

$$n = o(P_n).$$

Proof: From lemma 1 it follows that

$$\sum_{k=1}^{n} (n-k) \mid \Delta^2 p_k \mid = o(P_n),$$

from which, by taking the case $k = 1$, the result can easily be deduced.

Lemma 3. If $\sum_{k=1}^{n} k\,|\Delta p_k| = O(P_n)$, then $n\,|p_n| = O(P_n)$.

Proof:
$$P_k = (k+1)p_k + \sum_{j=1}^{k} j(p_{j-1}-p_j)$$

$$\therefore \qquad (k+1)p_k = P_k + \sum_{j=1}^{k} j(p_j - p_{j-1})$$

$$(n+1)p_n = P_n + \sum_{j=1}^{n} j(\Delta p_j) = O\left\{|P_n| + \sum_{j=1}^{n} j\,|\Delta p_j|\right\} = O(P_n).$$

Hence
$$n\,|p_n| = O(P_n).$$

Lemma 4. If $\sum_{k=1}^{n} |\Delta p_k| = O\left(\frac{P_n}{n}\right)$, then $\sum_{k=1}^{n} p_k(-1)^k = o(P_n)$.

Proof: See Hille and Tamarkin (1932).

Lemma 5. If $\sum_{k=1}^{n} |\Delta p_k| = O\left(\frac{P_n}{n}\right)$ and $\sum_{k=1}^{n} k\,|\Delta^2 p_k| = O\left(\frac{P_n}{n}\right)$,

then
$$n^2\,|\Delta p_n| = O(P_n).$$

Proof:
$$n^2 \Delta p_n = n \sum_{k=0}^{n-1} (\Delta p_k) + n \sum_{k=1}^{n} k(\Delta^2 p_k)$$

$$\therefore \qquad n^2\,|\Delta p_n| = O\left\{n \sum_{k=0}^{n-1} |\Delta p_k| + n \sum_{k=1}^{n} k\,|\Delta^2 p_k|\right\}$$
$$= O(P_n).$$

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