BIANCHI'S CHARACTERISTIC FUNCTION IN A CONGRUENCE OF RIBAUCOUR.

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The object of this paper is to express Bianchi's characteristic function in terms of infinitesimal elements of the director surface and the surface of reference in a congruence of Ribaucour. It has also been expressed in terms of the distances of focal points from the surface of reference. Some new theorems have been established with the help of these results.

1. A Congruence of Ribaucour is the congruence formed by rays through points on one surface parallel to the normals to another surface, the two surfaces corresponding with orthogonality of linear elements [1]. The former surface is called the surface of reference and the latter is known as the director surface.

Let x^i and y^i be the co-ordinates of the corresponding points on the director surface and the surface of reference respectively, then we must have,

$$dx^i \, . \, dy^i = 0 \quad (i, j = 1, 2, 3)$$

or

$$\left(\frac{\partial x^i}{\partial u^\alpha} \cdot \frac{\partial y^j}{\partial u^\beta} + \frac{\partial x^i}{\partial u^\beta} \cdot \frac{\partial y^j}{\partial u^\alpha}\right)\epsilon_{ij} = 0, \ \epsilon_{ij} = \delta^i_j . \quad (\alpha, \beta = 1, 2) \quad \ldots \quad (1.1)$$

If $g_{\alpha\beta}$, $d_{\alpha\beta}$ are the first and second fundamental tensors for the director surface, and

$$e_{\alpha\beta} \, e_{\gamma\delta} = g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma} \quad \ldots \quad \ldots \quad \ldots \quad (1.2)$$

then equation (1.1) may be replaced by an equivalent equation

$$\epsilon_{ij} = \frac{\partial x^i}{\partial u^\alpha} \cdot \frac{\partial y^j}{\partial u^\beta} = \phi e_{\alpha\beta} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.3)$$

The function 'ϕ' is known as Bianchi's characteristic function.

If X^i are the direction cosines of a normal to the surface at x^i then we have [2]

$$p_\gamma \equiv X^i \cdot y^i{}_{,\gamma} = E^{\alpha\beta} d_{\gamma\beta} \cdot \phi_{,\alpha} \quad \ldots \quad \ldots \quad \ldots \quad (1.4)$$

where $E^{\alpha\beta}$ is a tensor conjugate to $E_{\alpha\beta}$, and

$$E_{\alpha\beta} \cdot E_{\gamma\delta} = G_{\alpha\gamma} G_{\beta\delta} - G_{\alpha\delta} G_{\beta\gamma},$$

$G_{\alpha\beta}$ being the covariant components of the fundamental tensor for the spherical representation of the director surface.

The first fundamental tensor for the surface of reference is given by, [3]

$$a_{\alpha\beta} = g_{\alpha\beta} \phi^2 + p_\alpha p_\beta . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.5)$$

If

$$\bar{e}_{\alpha\beta} \bar{e}_{\gamma\delta} = a_{\alpha\gamma} a_{\beta\delta} - a_{\alpha\delta} a_{\beta\gamma}$$

$$\left[\bar{e}_{\alpha\beta} e^{\alpha\beta}\right]^2 = \phi^4 + p_\alpha p_\beta g^{\alpha\beta} \phi^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.6)$$

$$= \phi^4 + p_\alpha p_\beta g_{\delta\gamma} e^{\alpha\gamma} e^{\beta\delta} \phi^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.7)$$

or $$\left[\bar{e}_{\alpha\beta}e^{\alpha\beta}\right]^2 = \phi^4+p_\alpha p_\beta e^{\alpha\gamma}e^{\beta\delta}\left(a_{\delta\gamma}-p_\gamma p_\delta\right)$$

or $$\left[\bar{e}_{\alpha\beta}e^{\alpha\beta}\right]^2 = \phi^4+p_\alpha p_\beta e^{\alpha\gamma}e^{\beta\delta}a_{\delta\gamma} \text{ since } p_\alpha p_\beta p_\gamma p_\delta e^{\alpha\gamma}e^{\beta\delta} = 0$$

or $$\left[\bar{e}_{\alpha\beta}e^{\alpha\beta}\right]^2 = \phi^4+\left(\bar{e}_{\alpha\beta}e^{\alpha\beta}\right)^2 p_\alpha p_\beta \bar{e}^{\alpha\gamma}\,\bar{e}^{\beta\delta}\,a_{\gamma\delta} \quad \ldots \quad (1.8)$$

But $$p_\alpha p_\beta\,\bar{e}^{\alpha\gamma}\,\bar{e}^{\beta\delta}a_{\gamma\delta} = \sin^2\theta \quad \ldots \quad (1.9)$$

where 'θ' is the angle which a ray of the congruence makes with the normal to the surface of reference.

Hence $$\bar{e}_{\alpha\beta}\,e^{\alpha\beta}\cos\theta = \phi^2$$

or $$\frac{ds_1}{ds}\cos\theta = \phi^2 \quad \ldots \quad (1.10)$$

where ds_1, and ds are the areas of small elements on the surface of reference and the director surface.

Thus we see that Bianchi's characteristic function is denoted by the equation (1.10)

By Strazzeri's formula [4],

$$\frac{ds_1}{d\sigma}\cos\theta = \rho_1\rho_2 \quad \ldots \quad (1.11)$$

where ρ_1 and ρ_2 are the distances of the focal points from the surface of reference and $d\sigma$ is the element of area of the spherical representation of the congruence.

Dividing equation (1.10) by the equation (1.11) we get,

$$\frac{d\sigma}{ds} = \frac{\phi^2}{\rho_1\rho_2}$$

But by the theorem of Gauss

$$\frac{d\sigma}{ds} = k,$$

the total curvature of the director surface.

$$\therefore\ \phi^2 = k\rho_1\rho_2 = -k\rho^2 \text{ where } \rho_1 = -\rho_2 = -\rho \text{ (say)} \quad \ldots \quad (1.12)$$

Thus *Bianchi's characteristic function can also be expressed in terms of total curvature of director surface and the distances of focal points from the surface of reference.*

In particular, if we consider Isotropic congruences, then the director surface is a unit sphere, whose total curvature is unity.

$$\therefore\ \phi^2 = -\rho^2$$

But 'ϕ' can be shown to be the factor of proportionality between the corresponding coefficients of Sannia's two quadratic forms and hence is real

$$\therefore\ \rho \text{ is imaginary,}$$

i.e. *the focal points of an Isotropic congruence are imaginary. When the focal points coincide with the middle point, i.e. when the congruence is normal, Bianchi's characteristic function vanishes and the middle surface reduces to a point.*

From the formula (1.10) it follows that if ϕ vanishes, $ds_1 = 0$, i.e. the congruence consists of rays through a point.

2. The equation (1.9) gives an expression for $\sin^2\theta$ in terms of the covariant components of the fundamental tensor for the surface of reference. We shall now establish an expression for θ in terms of the covariant components of the fundamental tensor for the director surface.

From the equations (1.7), (1.8) and (1.9) we get

$$p_\alpha p_\beta g_{\delta\gamma} e^{\alpha\gamma} e^{\beta\delta} \, . \, \phi^2 = \left[\bar{e}_{\alpha\beta} e^{\alpha\beta}\right]^2 \sin^2\theta$$

or

$$p_\alpha p_\beta g^{\alpha\beta} = \sin^2\theta \left(\phi^2 + p_\alpha p_\beta g^{\alpha\beta}\right) \text{ by virtue of (1.6)}$$

$$\therefore \tan^2\theta = \frac{1}{\phi^2} p_\alpha p_\beta g^{\alpha\beta} \, . \qquad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

When expanded this equation becomes

$$\tan^2\theta = \frac{(p_2)^2 g_{11} + (p_1)^2 g_{22} - 2 g_{12} p_1 p_2}{\phi^2 (g_{11} g_{22} - g_{12}{}^2)}$$

Another expression for $\tan\theta$ can be obtained by putting the values of p_γ from (1.4)

$$\tan^2\theta = \frac{1}{\phi^2}\left[E^{\gamma\delta} E^{\rho\nu} \phi_{,\gamma} \phi_{,\rho} d_{\alpha\delta} d_{\beta\nu} g^{\alpha\beta}\right]$$

But

$$G_{\alpha\beta} = d_{\alpha\delta} d_{\beta\nu} g^{\delta\nu} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

$\therefore$ The equation (2.2) becomes

$$\tan^2\theta = \frac{1}{\phi^2} E^{\gamma\delta} E^{\rho\nu} \phi_{,\gamma} \phi_{,\rho} G_{\delta\nu}$$

or

$$\tan^2\theta = \frac{1}{\phi^2} \phi_{,\gamma} \phi_{,\rho} G^{\gamma\rho} \, .$$

When expanded this equation becomes

$$\tan^2\theta = \frac{\left(\frac{\partial\phi}{\partial u^2}\right)^2 G_{11} + \left(\frac{\partial\phi}{\partial u^1}\right)^2 G_{22} - 2\frac{\partial\phi}{\partial u^1} \cdot \frac{\partial\phi}{\partial u^2} \cdot G_{12}}{\phi^2 (G_{11} G_{22} - G_{12}{}^2)}$$

If Y^i are the direction cosines of the normal to the surface of reference

then

$$Y^i = z_i e_{\alpha\beta} \, \bar{e}^{\,\alpha\beta} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

where z^i are the co-ordinates of the corresponding points on the surface S_0 associated to the director surface.

From equation (1.10) this equation becomes

$$Y^i = z^i \cos\theta . \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

If $\theta = 0$ the congruence consists of normals to the surface of reference,

$$Y^i = z^i .$$

Hence in a congruence of Ribaucour, formed by normals to the surface of reference, the surface associated to the director surface is a sphere. Conversely, if the surface S_0 is a sphere, then from (3.2).

$$Y^i = z^i \cos\theta$$

$$Y^i Y^i = z^i z^i \cos^2\theta$$

$$\text{or } \cos\theta = 1 \text{ or } \theta = 0.$$

Hence *if the surface associated to the director surface of a congruence of Ribaucour is a sphere, the congruence consists of rays normal to the surface of reference.*

4. From (1.5), the null lines on the surface of reference are given by

$$a_{\alpha\beta}du^\alpha du^\beta \equiv \phi^2 g_{\alpha\beta}du^\alpha du^\beta + p_\alpha p_\beta du^\alpha du^\beta = 0. \quad .. \quad .. \quad (4.1)$$

Thus the null lines on the surface of reference correspond to the null lines on the director surface if

$$p_\alpha p_\beta du^\alpha du^\beta = 0$$

or

$$(p_\alpha du^\alpha)^2 = 0,$$

which is true if the congruence is formed by normals to the surface of reference. Hence *if a congruence of Ribaucour is formed by normals to the surface of reference, the null lines on the surface of reference and the director surface correspond.*

From the equation (1.3) the relation between the director surface and the surface of reference is reciprocal. Therefore the equation (1.3) can also be written as

$$\epsilon_{ij}\frac{\partial x^i}{\partial u^\alpha}\cdot\frac{\partial y^j}{\partial u^\beta} = \phi_1 \bar{e}_{\alpha\beta} \quad .. \quad .. \quad .. \quad .. \quad (4.2)$$

where ϕ_1 also is 'Bianchi's characteristic function,' but with respect to $\bar{e}_{12}$ of the surface of reference.

From the equations (1.3) and (4.1)

$$\phi_1 \bar{e}_{\alpha\beta} = \phi e_{\alpha\beta}$$

or

$$\frac{\bar{e}_{\alpha\beta}}{e_{\alpha\beta}} = \frac{\phi}{\phi_1} \text{ or } \frac{ds_1}{ds} = \frac{\phi}{\phi_1}.$$

Using equation (1.10)

$$\phi\phi_1 = \cos\theta. \quad .. \quad .. \quad .. \quad .. \quad (4.3)$$

i.e. *the cosine of the angle between a ray of a congruence of Ribaucour and the normal to the surface of reference at the point where the ray cuts it is the product of Bianchi's two characteristic functions.*

REFERENCES.

1. Eisenhart: Differential Geometry of curves and surfaces, p. 420.
2. Eisenhart: loc. cit., p. 375.
3. Eisenhart, loc. cit., p. 390, ex. 4.
4. Strazzeri, sulle Congruenze di retti, Rendiconti del circolo Matematico di Palermo, 1927, p. 138.
5. Eisenhart, loc. cit., p. 382.