# ON NUMBERS WHICH CAN BE EXPRESSED BY A GIVEN FORM.

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In their paper ' On numbers which can be expressed as a sum of two squares ' Bambah and Chowla (1947) have established the following theorem (first obtained by Dr. Vijayaraghavan) :—

Let $\epsilon$ denote an arbitrary positive number. Then there exists between $x$ and $x+2\sqrt{2+\epsilon}\, x^{1/4}$ an integer which can be expressed as a sum of two squares (of integers) for all $x > x_0(\epsilon)$.

Using more straightforward arguments one can, however, establish the following :—

THEOREM 1. Between $x$ and $x+2\sqrt{2}\, x^{1/4}+1$ there exists at least one integer which can be expressed as a sum of two squares of integers for all $x \geq 0$.

The proof of this theorem will be clear from the sequel.

The object of the present paper is to study a similar but more general problem. More precisely, we try to find a function $f(x)$ such that between $x$ and $x+f(x)$ there lies at least one number expressible by means of a given function with its arguments taking integral values. In this paper we take this function to be

$$\alpha_1 x_1^{\theta_1}+\alpha_2 x_2^{\theta_2}+ \ldots +\alpha_n x_n^{\theta_n} \text{ where } \theta_i > 1,\ \alpha_i > 0.$$

Then we show that

$$f(x) = O(x^{(1-1/\theta_1)(1-1/\theta_2)\ldots(1-1/\theta_n)}).$$

Bambah and Chowla's reasonings, it appears, cannot be generalized easily to obtain this result.

THEOREM 2. Given any arbitrary form

$$\alpha_1 x_1^{\theta_1}+\alpha_2 x_2^{\theta_2} \ldots +\alpha_n x_n^{\theta_n}$$

with $\alpha_i > 0$, $\theta_i > 1$, then for any positive $\epsilon$ there exists between $x$ and

$$x+Cx^{(1-1/\theta_1)(1-1/\theta_2)\ldots(1-1/\theta_n)}$$

where $$C = C_0(1+\epsilon), C_0 = C_0(\alpha_1, \alpha_2 \ldots \alpha_n, \theta_1, \theta_2, \ldots \theta_n), x > x_0(\epsilon),$$

at least one number which can be expressed by means of the given form with its $x$'s taking integral values.

Given $x$, choose the unique integer $x_1$ defined by

$$\left(\frac{x}{\alpha_1}\right)^{1/\theta_1} = x_1+\epsilon_1, \qquad 0<\epsilon_1<1.$$

In other words we take $x_1$ to be the greatest integer contained in

$$\left(\frac{x}{\alpha_1}\right)^{1/\theta_1}.$$

Then

$$\alpha_1 x_1^{\theta_1} = \alpha_1 \left\{ \left(\frac{x}{\alpha_1}\right)^{1/\theta_1} - \epsilon_1 \right\}^{\theta_1} = x-\rho_1 x^{(1-1/\theta_1)} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where $$0 < \rho_1(x) < \theta_1 \alpha_1^{1/\theta_1} \text{ provided } x > \text{some } x'.$$

Of course $$\overline{Lt}\rho_1(x) = \theta_1 \alpha_1^{1/\theta_1}.$$

Next take the unique integer $x_2$ given by

$$\left\{\frac{\rho_1 x^{(1-1/\theta_1)}}{\alpha_2}\right\}^{1/\theta_2} = x_2 + \epsilon_2,\ 0 < \epsilon_2 < 1.$$

Then $$\alpha_2 x_2^{\theta_2} = \rho_1 x^{(1-1/\theta_1)} - \rho_2 x^{(1-1/\theta_1)(1-1/\theta_2)} \quad \ldots \quad (2)$$

where $0 < \rho_2(x) < \theta_2 \theta_1^{(1-1/\theta_2)} \alpha_1^{(1-1/\theta_2)\cdot 1/\theta_1} \alpha_2^{1/\theta_2}$

provided $x > x'' > x'$. Clearly $\overline{Lt}\rho_2(x) = \theta_2\theta_1^{(1-1/\theta_2)}\alpha_1^{(1-1/\theta_2)1/\theta_1}\alpha_2^{1/\theta_2}$.

Similarly take $x_3 = \left[\frac{\rho_2 x^{(1-1/\theta_1)(1-1/\theta_2)}}{\alpha_3}\right]^{1/\theta_3}$ and proceed in this way up to $x_{n-1}$.

Then $$\alpha_{n-1} x_{n-1}^{\theta_{n-1}} = \rho_{n-2} x^{(1-1/\theta_1)\ldots(1-1/\theta_{n-2})} - \rho_{n-1} x^{(1-1/\theta_1)\ldots(1-1/\theta_{n-1})}.$$

Finally choose as $x_n$ the integer just exceeding

$$\left\{\frac{\rho_{n-1}}{\alpha_n} x^{(1-1/\theta_1)\ldots\ldots(1-1/\theta_{n-1})}\right\}^{1/\theta_n} \text{ so that}$$

$$x_n = \left\{\frac{\rho_{n-1}}{\alpha_n} x^{(1-1/\theta_1)\ldots\ldots(1-1/\theta_{n-1})}\right\}^{1/\theta_n} + \epsilon_n,\ 0 < \epsilon_n < 1$$

and hence

$$\alpha_n x_n^{\theta_n} = \rho_{n-1} x^{(1-1/\theta_1)\ldots\ldots(1-1/\theta_{n-1})} \left[1 + \theta_n \epsilon_n \left\{\frac{\rho_{n-1}}{\alpha_n}\cdot x^{(1-1/\theta_1)\ldots\ldots(1-1/\theta_{n-1})}\right\}^{-1/\theta_n}(1+\epsilon)\right]$$

$$< \rho_{n-1} x^{(1-1/\theta_1)\ldots\ldots(1-1/\theta_{n-1})} + C_0(1+\epsilon) x^{(1-1/\theta_1)\ldots\ldots(1-1/\theta_n)} \quad \ldots \quad (n)$$

where $$C_0 = \theta_n \theta_{n-1}^{(1-1/\theta_n)} \text{ etc. and } \epsilon \to 0 \text{ as } x \to \infty.$$

Adding (1), (2), .... ($n$) we get

$$x < \alpha_1 x_1^{\theta_1} + \alpha_2 x_2^{\theta_2} \ldots\ldots + \alpha_n x_n^{\theta_n} < x + C_0(1+\epsilon) x^{(1-1/\theta_1)\ldots\ldots(1-1/\theta_n)}$$

whenever $$x > x_0(\alpha_1 \alpha_2 \ldots\ldots \alpha_n, \theta_1, \theta_2, \ldots\ldots \theta_n, \epsilon).$$

Making the above selections in all possible different orders one can obtain the minimum value for $C_0$. In the above proof we have not taken any $\epsilon_i = 0$, for in that case $x$ itself becomes representable by the form.

If in the above theorem we take in particular

$$\alpha_1 = \alpha_2 \ldots\ldots = \alpha_k = 1,\ \theta_1 = \theta_2 = \theta_3 \ldots\ldots = \theta_k = k$$

we get the following theorem.

THEOREM 3. Between $x$ and $x + (1+\epsilon)Cx^{(1-1/k)^k}$, where $C$ is a constant, $\epsilon > 0$, $x > x_0(\epsilon, k)$, there always exists an integer which can be expressed as a sum $k$ integral $k$th powers.

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## REFERENCE.

Bambah, R. P. and Chowla, S. (1947). On numbers which can be expressed as a sum of two squares. *Proc. Nat. Inst. of Sc. of India*, **13**, 101–103.