ON A NEW TYPE OF PARTITION.

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Abstract.

The present paper is concerned with an investigation of a new type of partitions in which the total number of summands is unrestricted but the number of different summands that can occur is only N. The problem is treated analytically and asymptotic formulae have been derived in the two limiting cases: (*i*) when N is very large and (*ii*) when it is small. The case when N is unrestricted is also dealt with.

1. *Introduction.*

An interesting application (Auluck, Kothari, Agarwala, 1946-47) of the methods of statistical thermodynamics has recently been made to the problem of the partitions of numbers. A thermodynamic investigation of the partition theory suggests new types of problems and draws attention to various generalizations of the partition concept. The problem of partitions corresponds physically to an evaluation of the number of states accessible to a thermodynamic assembly suitable for a particular partition function. If, for example, the problem be to evaluate the partitions of an integer n into smaller positive integers, no restriction being placed on either the number of summands or the repetition of any summand, the suitable thermodynamic assembly is one of linear Bose-Einstein oscillators in which the individual energy levels are equidistant and given by $\epsilon_i = i\Delta$ where $i = 1, 2, 3, \ldots\ldots$ and Δ is the spacing of the levels. The required partitions of n then correspond to an evaluation of the accessible states of this assembly when in the energy state $E = n\Delta$. This number, as is well known, is given by the coefficient of x^n in the development

$$Z = \prod_{i=1}^{\infty} (1 + x^i + x^{2i} + x^{3i} + \ldots\ldots\ldots),$$

where $x = e^{-\Delta/kT}$.

By a suitable modification of this assembly one can impose restrictions on the number and nature of the summands. Thus if in the above example we require partitions into a fixed number k of summands the number of oscillators in the assembly must be restricted to k and the partitions are then enumerated by the coefficient of $x^n \zeta^k$ in the expansion

$$Z = \prod_{i=1}^{\infty} (1 + \zeta x^i + \zeta^2 x^{2i} + \zeta^3 x^{3i} + \ldots\ldots\ldots).$$

If in addition to integers we also require half odd integers to occur in the partitions of n and allow any summand to repeat only an odd number of times the states-function of the corresponding thermodynamic assembly will be given by

$$Z = \prod_{i=1}^{\infty} (1 + \zeta x^{\frac{1}{2}i} + \zeta^3 x^{\frac{3}{2}i} + \zeta^5 x^{\frac{5}{2}i} + \ldots\ldots\ldots), \qquad \ldots \qquad \ldots \quad (1)$$

and the partitions into a fixed number of summands k will again be enumerated by the coefficient of $\zeta^k x^n$ in the above expansion. What restricts the total number of summands is obviously the power of ζ.

An interesting problem arises when we consider the generating function *

$$Z = \prod_{i=1}^{\infty} (1+\zeta x^{\frac{1}{2}i}+\zeta x^{\frac{3}{2}i}+\zeta x^{\frac{5}{2}i}+\ldots\ldots\ldots). \qquad \ldots \qquad \ldots \qquad (2)$$

Since the powers of x in (2) are the same as in (1) the generating function (2) will still enumerate partitions into integers and half odd integers repeated only an odd number of times with one important difference that while the power k of ζ occurring in (1) restricts the total number of summands to a fixed number k this power in (2) restricts not the total number of summands but the number of different † summands that can occur. Such partitions, we believe, have not been considered before and it is, therefore, of interest to investigate the problem in detail.

In section 2 we give a general expression for the number of accessible states of the Born-Peng assembly. In section 3 we evaluate the partitions of n in the above specified manner when the number N of the types of summands that can occur in large and in section 4 the partitions are evaluated for small N. These are the two limiting cases. In section 5 we solve the same problem for $\zeta = 1$ which removes the restriction on the types and, therefore, enumerates the partitions of n into integers and half odd integers repeated only an odd number of times.

2. *The density $\rho(E, N)$ of energy levels.*

In the customary theory of Born and Peng the energy levels ϵ_i of the 'radiation' oscillators are assumed to be half odd integral and capable of accommodating 1, 2, 3, ... quanta. For purposes of interpretation in the partition theory we shall assume that in the states-function (2), which when written out in detail is

$$(1+\zeta x^{\frac{1}{2}}+\zeta x^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}}+\zeta x^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}}+\ldots).$$
$$(1+\zeta x^{1}+\zeta x^{1+1+1}+\zeta x^{1+1+1+1+1}+\ldots)$$
$$(1+\zeta x^{\frac{3}{2}}+\zeta x^{\frac{3}{2}+\frac{3}{2}+\frac{3}{2}}+\zeta x^{\frac{3}{2}+\frac{3}{2}+\frac{3}{2}+\frac{3}{2}+\frac{3}{2}}+\ldots),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

the energy levels are $\frac{1}{2}$, 1, $\frac{3}{2}$, 2 ... and each of these levels can admit only an odd number of quanta, i.e., we assume

$$\epsilon_i = i\Delta\,(i = 1, 2, 3, \ldots) \text{ and } \Delta = \tfrac{1}{2}.$$

To evaluate the required partitions of n we have then to evaluate the coefficient of $\zeta^N x^n$ in the expansion (2). This is the same as evaluating first the coefficient of $\zeta^N x^E$ $(E = n\Delta)$ in the development

$$\prod_{i=1,2,3,\ldots} (1+\zeta x^{\frac{1}{2}\epsilon_i}+\zeta x^{\frac{3}{2}\epsilon_i}+\zeta x^{\frac{5}{2}\epsilon_i}+\ldots), \qquad \ldots \qquad \ldots \qquad (3)$$

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* This function arises in the recent theory (Schrodinger, 1946) of Born and Peng in which these authors in an attempt to get rid of the infinite self-energy of the radiation field attribute to any one of the 'hohlraum' oscillators two fundamentally different situations. It can either be not excited at all when it has the energy zero or excited when it has one of the energies $\frac{1}{2}h\nu_s$, $\frac{3}{2}h\nu_s$, $\frac{5}{2}h\nu_s$, ..., where ν_s is the frequency of the sth oscillator. It is further assumed that the number of excited oscillators is only N. The 'Zustände' of such an assembly is given by (2).

† Consider, for instance, the number 8 partitioned in the following manner :—

$$8 = \tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+1+1+1+\tfrac{3}{2}+2.$$

In the generating function (1) the above partitions of (8) would correspond to the coefficient of ζ^8 while in (2) where the types of summands are restricted and not the total number, these partitions will correspond to the coefficient of ζ^4.

and then putting $\frac{E}{\Delta} = n$. The coefficient of $\zeta^N x^E$ in (3) is given by

$$\rho(E, N) = \frac{1}{(2\pi i)^2} \oint\oint \frac{\underset{\epsilon_i}{\Pi}\left\{1+\frac{\zeta}{x^{-\frac{1}{2}\epsilon_i}-x^{\frac{1}{2}\epsilon_i}}\right\}}{\zeta^{N+1}x^{E+1}}\, dx\, d\zeta$$

$$= \frac{1}{(2\pi i)^2} \oint\oint \frac{d\zeta}{\zeta}\frac{dx}{x}\,\Phi(\zeta, x), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

where the integrals are taken over closed contours round the origins of the complex ζ and x planes and

$$\left.\begin{aligned} \Phi(\zeta, x) &= \frac{e^{f(\zeta, x)}}{\zeta^N x^E}, \\ \text{with} \qquad f(\zeta, x) &= \sum_{\epsilon_i} \log\left\{1+\frac{\zeta}{x^{-\frac{1}{2}\epsilon_i}-x^{\frac{1}{2}\epsilon_i}}\right\}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (5)$$

The integrand has along the real x and ζ axes sharp minima at $\zeta = a$ and $x = b$ where a and b are given by

$$a\frac{\partial f}{\partial a} = N, \quad b\frac{\partial f}{\partial b} = E. \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

By letting the paths of integration pass through these points one can apply the usual method of steepest descents for the evaluation of (4). The result (Lier *and* Uhlenbuk, 1937) is

$$\rho(E, N) = \frac{1}{2\pi\sqrt{k_0k_2-k_1^2}} \exp\,\{\alpha N+2\beta E+f(a, b)\}, \quad \ldots \quad \ldots \quad (7)$$

where

$$\left.\begin{aligned} a^{-1} &= e^{\alpha} \\ b^{-1} &= e^{2\beta} \end{aligned}\right\}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

and

$$\left.\begin{aligned} k_0 &= a\frac{\partial f}{\partial a}+a^2\frac{\partial^2 f}{\partial a^2}, \\ k_1 &= ab\frac{\partial^2 f}{\partial a \partial b}, \\ k_2 &= b\frac{\partial f}{\partial b}+b^2\frac{\partial^2 f}{\partial b^2}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

3. *Calculation of* ρ (E, N) *when* N *is very large.*

From (5), using (8) we have

$$f(\alpha, \beta) = \sum_{\epsilon=1}^{\infty} \log\left\{1+\frac{1}{e^{\alpha}(e^{\beta\epsilon}-e^{-\beta\epsilon})}\right\}$$

$$= -\sum_{\epsilon=1}^{\infty}\log\left(1-e^{-2\beta\epsilon}\right)+\sum_{\epsilon=1}^{\infty}\log\left(1+e^{-\alpha-\beta\epsilon}\right)$$

$$+\sum_{\epsilon=1}^{\infty}\log\left(1-\frac{e^{-2\beta\epsilon}}{1+e^{-\alpha-\beta\epsilon}}\right). \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

Now

$$-\sum_{\epsilon=1}^{\infty} \log \left(1-e^{-2\beta\epsilon}\right) = \frac{\pi^2}{12\beta}+\tfrac{1}{2}\log 2\beta-\tfrac{1}{2}\log(2\pi)+0(\beta). \quad .. \quad (11)$$

Also

$$\sum_{\epsilon=1}^{\infty} \log \left(1+e^{-\alpha-\beta\epsilon}\right) = \sum_{\epsilon=0}^{\infty} \log\left(1+e^{-\alpha-\beta\epsilon}\right)-\log\left(1+e^{-\alpha}\right)$$

$$= \int_0^{\infty} \log\left(1+e^{-\alpha-\beta\epsilon}\right) d\epsilon - \tfrac{1}{2}\log\left(1+e^{-\alpha}\right)+\frac{\beta}{12}\frac{e^{-\alpha}}{1+e^{-\alpha}},$$

where we have replaced the summation by an integration using the Euler-Maclaurin formula. Consider the integral

$$\int_0^{\infty} \log\left(1+e^{-\alpha-\beta\epsilon}\right) d\epsilon = \beta \int_0^{\infty} \frac{\epsilon\, d\epsilon}{1+e^{\alpha+\beta\epsilon}} = \frac{\beta}{2}\int_0^{\infty} \frac{d}{d\epsilon}(\epsilon^2)\frac{d\epsilon}{1+e^{\alpha+\beta\epsilon}}.$$

Using Sommerfeld's lemma we have

$$\int_0^{\infty} \log\left(1+e^{-\alpha-\beta\epsilon}\right) d\epsilon \approx \frac{1}{2\beta}\left(\alpha^2+\frac{\pi^2}{3}\right),$$

so that

$$\sum_{\epsilon=1}^{\infty} \log\left(1+e^{-\alpha-\beta\epsilon}\right) \approx \frac{1}{2\beta}\left(\alpha^2+\frac{\pi^2}{3}\right)+\frac{\alpha}{2}+\frac{\beta}{12}. \quad .. \quad .. \quad (12)$$

Considering the last summation in (10) we have

$$\sum_{\epsilon=1}^{\infty} \log\left(1-\frac{e^{-2\beta\epsilon}}{1+e^{-\alpha-\beta\epsilon}}\right) = \sum_{\epsilon=1}^{\infty}\left\{-\frac{e^{-2\beta\epsilon}}{1+e^{-\alpha-\beta\epsilon}}-\tfrac{1}{2}\frac{e^{-4\beta\epsilon}}{(1+e^{-\alpha-\beta\epsilon})^2}-\dots\right\}. \quad (13)$$

Now

$$-\sum_{\epsilon=1}^{\infty}\frac{e^{-2\beta\epsilon}}{1+e^{-\alpha-\beta\epsilon}} = -\int_0^{\infty}\frac{e^{-2\beta\epsilon}}{1+e^{-\alpha-\beta\epsilon}}\,d\epsilon+0(e^{\alpha})$$

$$= -\frac{e^{\alpha}}{\beta}\left\{1+\alpha e^{\alpha}-e^{\alpha}+\frac{e^{2\alpha}}{2}\right\}+0(e^{\alpha}).$$

β is a very small quantity (see equation 18) and $\alpha \to -\infty$. If we neglect quantities of the order of $\frac{e^{\alpha}}{\beta}$ and less the left side of (13) tends to zero. We thus have from equations (10), (11) and (12)

$$f(\alpha,\beta) = \frac{1}{2\beta}\left(\alpha^2+\frac{\pi^2}{2}\right)+\frac{\alpha}{2}+\frac{\beta}{12}+\tfrac{1}{2}\log 2\beta-\tfrac{1}{2}\log 2\pi+0\left(\frac{e^{\alpha}}{\beta}\right). \quad .. \quad (14)$$

From (6) and (14) we have

$$\left.\begin{aligned} E &= b\frac{\partial f}{\partial b} = -\frac{1}{2}\frac{\partial f}{\partial \beta} = \frac{1}{4\beta^2}\left(\alpha^2+\frac{\pi^2}{2}\right)-\frac{1}{4\beta}-\frac{1}{24} \\ \text{and} \qquad N &= a\frac{\partial f}{\partial a} = -\frac{\partial f}{\partial \alpha} = -\frac{\alpha}{\beta}-\frac{1}{2}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (15)$$

It can be shown easily from (8), (9) and (14) that, to the desired order of approximation,

$$k_0k_2-k_1^2 = \frac{\pi^2}{8\beta^4}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

Also

$$\alpha N+2\beta E+f(\alpha, \beta) = \frac{\pi^2}{2\beta}-\frac{1}{2}\log\frac{\pi}{\beta}-\frac{1}{2}. \quad \ldots \quad \ldots \quad (17)$$

From (15) we have

$$\alpha \approx -\beta N,$$

so that

$$E = \frac{1}{4\beta^2}\left(\beta^2N^2+\frac{\pi^2}{2}\right)-\frac{1}{4\beta}-\frac{1}{24},$$

or

$$E-\frac{1}{4}N^2 \approx \frac{\pi^2}{8\beta^2},$$

giving

$$\beta = \frac{\pi^2}{2\sqrt{2\left(E-\frac{1}{4}N^2\right)}}. \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

The expression (7) for $\rho(E, N)$ becomes with the help of (16), (17) and (18)

$$\rho(E, N) = \frac{1}{4}\cdot\frac{1}{\left\{2\left(E-\frac{1}{4}N^2\right)\right\}^{\frac{5}{4}}} \exp\left\{\pi\sqrt{2\left(E-\frac{1}{4}N^2\right)}\right\}.$$

If Δ denotes the spacing of the individual energy levels (in our case $\Delta = \frac{1}{2}$) the number of partitions $P(n, N)$ of a large number

$$n = \frac{E}{\Delta} = \frac{E}{1/2}$$

into exactly N types is

$$P(n, N) = \frac{\Delta}{4\left\{\frac{E}{1/2}-\frac{1}{2}N^2\right\}^{\frac{5}{4}}} \exp\left\{\pi\sqrt{\frac{E}{1/2}-\frac{1}{2}N^2}\right\},$$

or

$$P(n, N) = \frac{1}{8(n-\frac{1}{2}N^2)^{\frac{5}{4}}} \exp\left\{\pi\sqrt{n-\tfrac{1}{2}N^2}\right\}. \quad \ldots \quad \ldots \quad (19)$$

The above formula holds only for large values of N and $n \gg \frac{1}{2}N^2$. If we plot $P(n, N)$ against N the maximum is expected to occur somewhere near $N \sim n^{\frac{1}{2}}$ and therefore the formula holds only at the tail end of the curve.

4. *Calculation of $\rho(E, N)$ for small N.*

We consider the case when α is large and positive. We have

$$f(a, b) = \sum_{\epsilon=1}^{\infty} \log \left(1+\frac{a}{b^{-\epsilon/2}-b^{\epsilon/2}}\right) = \sum_{\epsilon=1}^{\infty} \log \left(1+\frac{a}{2 \sinh \beta\epsilon}\right). \quad .. \quad (20)$$

Also

$$N = a\,\frac{\partial f}{\partial a} = a \sum_{\epsilon=1}^{\infty} \frac{1}{a+2 \sinh \beta\epsilon}$$

$$= \frac{a}{\beta}\int_0^\infty \frac{dx}{a+2\sinh x} - \frac{1}{2}+\frac{1}{6}\left(\frac{\beta}{a}\right)+0\left\{\left(\frac{\beta}{a}\right)^3\right\}.$$

Now for small values of a ($a \to 0$ since $a = e^{-\alpha}$) we have

$$\int_0^\infty \frac{dx}{a+2\sinh x} = \frac{1}{2}\log\left\{\frac{4-a}{a\left(1+\frac{a}{4}\right)}\cdot\frac{2+a}{2-a}\right\},$$

so that

$$N = \frac{e^{-\alpha}}{\beta}\left(\log 2+\frac{\alpha}{2}+\frac{e^{-\alpha}}{4}\right)-\frac{1}{2}+\frac{1}{6}\beta e^{\alpha}+0\{(\beta e^{\alpha})^3\}. \quad .. \quad (21)$$

We further have

$$E = b\,\frac{\partial f}{\partial b} = \frac{1}{2}\sum_{\epsilon=0}^{\infty}\frac{\epsilon \coth \beta\epsilon}{1+2e^{\alpha}\sinh\beta\epsilon}-\frac{1}{2}\beta = \frac{1}{2}\int_0^\infty \frac{\epsilon\coth\beta\epsilon \,.\, d\epsilon}{1+2e^{\alpha}\sinh\beta\epsilon}-\frac{1}{4\beta}+\frac{e^{\alpha}}{12}. \quad .. \quad (22)$$

Denoting the first term in (22) by I and putting $c = 2e^{\alpha}$ we have

$$I = \frac{1}{2\beta^2}\int_0^\infty \frac{x\coth x\,dx}{1+c\sinh x}.$$

Putting $\sinh x = z/c$, and integrating by parts we have

$$I = \int_0^\infty \frac{dz}{(c^2+z^2)^{1/2}}\log\frac{1+z}{z}\,dz,$$

or on putting $z = c\tan\theta$,

$$I = \int_0^{\pi/2}\log\left(1+\frac{\cot\theta}{c}\right)\sec\theta\,d\theta$$

$$= \int_0^{\lambda}(-\log c-\log\tan\theta+c\tan\theta-\tfrac{1}{2}c^2\tan^2\theta+\tfrac{1}{3}c^3\tan^3\theta\ldots)\sec\theta\,d\theta$$

$$+\int_{\lambda}^{\pi/2}\left(\frac{\cot\theta}{c}-\frac{\cot^2\theta}{2c^2}+\frac{\cot^3\theta}{3c^3}\ldots\right)\sec\theta\,d\theta,$$

where $\tan\lambda = 1/c$.

Performing the integrations term by term we have

$$I = (\sqrt{1+c^2}-c)+\left(\frac{c^2}{4}\log\frac{1+\sqrt{1+c^2}}{c}-\frac{1}{4}\sqrt{1+c^2}\right)+\frac{2c^3}{9}+\frac{1}{9}(1+c^2)^{\frac{3}{2}}$$

$$-\frac{c^2}{3}(1+c^2)^{\frac{1}{2}}+\frac{1}{c}\log\left(c+\sqrt{1+c^2}\right)+\frac{1}{2c^2}\left(1-\sqrt{1+c^2}\right)$$

$$+\frac{1}{6c^3}\left\{c\sqrt{1+c^2}-\log\left(c+\sqrt{1+c^2}\right)\right\}-\frac{1}{12c^4}\left\{2-(1+c^2)^{\frac{1}{2}}(2-c^2)\right\}$$

$$+\frac{1}{c}-\frac{1}{2c^3}\cdot\frac{1}{3^2}+\frac{1.3}{2.4}\frac{1}{5^2}\cdot\frac{1}{c^5}-\frac{1.3.5}{2.4.6}\cdot\frac{1}{7^2}\cdot\frac{1}{c^7}-\ldots.$$

Hence the expression for E becomes

$$E = \frac{e^{-\alpha}}{4\beta^2}\left\{2\log 2+\alpha+\frac{11}{12}+\frac{1}{3}\beta^2e^{2\alpha}-\beta e^{\alpha}\right\}, \qquad \ldots \quad \ldots \quad (23)$$

for values of $c \equiv 2e^{\alpha} \gg 1$.

β is a very small quantity and if we neglect quantities of the order of βe^{α}, (21) and (23) become

$$N = \frac{e^{-\alpha}}{\beta}\left(\frac{\alpha}{2}+\log 2\right),$$

and

$$E = \frac{e^{-\alpha}}{4\beta^2}\left(\alpha+2\log 2+\frac{11}{12}\right).$$

For large α

$$\left.\begin{aligned} N &\approx \frac{e^{-\alpha}}{2\beta}\cdot\alpha, \\ E &\approx \frac{e^{-\alpha}}{4\beta^2}\cdot\alpha. \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

From (6) and (24) we obtain

$$f(\alpha, \beta) \approx \frac{\alpha e^{-\alpha}}{2\beta} = 2\beta E,$$

so that

$$\alpha N+2\beta E+f(\alpha, \beta) \approx \alpha N+4\beta E. \qquad \ldots \quad \ldots \quad \ldots \quad (25)$$

Also

$$\beta \approx \frac{N}{2E}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

and therefore

$$\alpha \approx \log\frac{E}{N^2}+\log\log\frac{E}{N^2}. \qquad \ldots \quad \ldots \quad \ldots \quad (27)$$

Using (27) in (25) we get

$$\alpha N+2\beta E+f(\alpha,\beta) \approx N\left(\log\frac{E}{N^2}+\log\log\frac{E}{N^2}+2\right) \approx N\log\frac{E}{N^2}+2N. \quad \ldots \quad (28)$$

It may easily be shown that

$$k_0k_2-k_1^2 \approx \frac{\alpha^2e^{-2\alpha}}{16\beta^4} \approx E^2. \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

Substituting (28) and (29) in (7) we get

$$\rho(E, N) \approx \frac{1}{2\pi}\frac{E^{N-1}}{N^{2N}}e^{2N}.$$

Since the spacing of the levels is $\frac{1}{2}$ (energy units) we have

$$P(n, N) \approx \frac{1}{2\pi}\frac{n^{N-1}}{2^N N^{2N}}e^{2N}. \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

Expression (30) is valid for values of $N \ll n^{\frac{1}{3}}$, i.e., on the left of the maximum of the $P(n, N)$, N curve.

4. *Partitions when the types of summands are unrestricted :*

In this section we shall consider the case when there is no restriction on the types of summands that can occur. This is achieved by putting $\zeta = 1$ in the generating function. We shall in what follows consider only the partitions into integers repeated an odd number of times, since the partitions of $2n$ into integers can be looked upon as the partitions of n into integers and half odd integers. The generating function to be considered is then

$$f(x) = \prod_{i=1}^{\infty}(1+x^i+x^{3i}+x^{5i}+\ldots\ldots)$$

$$= \left(1+\frac{x}{1-x^2}\right)\left(1+\frac{x^2}{1-x^4}\right)\left(1+\frac{x^3}{1-x^6}\right)\ldots\ldots$$

$$= \frac{(1+x-x^2)(1+x^2-x^4)(1+x^3-x^6)\ldots\ldots}{(1-x^2)(1-x^4)(1-x^6)\ldots\ldots}. \quad \ldots \quad \ldots \quad (31)$$

We have to find the coefficient of x^n in (31). If this coefficient is denoted by $\omega(n)$ we have * as usual

$$\omega(n) = \frac{e^{z(\mu)}}{\left(-2\pi\frac{\partial n}{\partial\mu}\right)^{\frac{1}{2}}}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

where

$$x = e^{-\mu},\ n = -\frac{\partial}{\partial\mu}\log f(e^{-\mu}),\ z(\mu) = n\mu+\log f(e^{-\mu}).$$

Now

$$\log f(e^{-\mu}) = \sum_{r=1}^{\infty}\log\left(1+e^{-r\mu}-e^{-2r\mu}\right)-\sum_{r=1}^{\infty}\log\left(1-e^{-2r\mu}\right). \quad \ldots \quad (33)$$

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* For a proof see Auluck, F. C., and Agarwala, B. K., *loc. cit.*

But

$$\prod_{r=1}^{\infty} \frac{1}{1-e^{-2r\mu}} = \frac{e^{\pi^2/12\mu}}{\sqrt{\pi}} \sqrt{\mu}.$$

Therefore

$$-\sum_{r=1}^{\infty} \log\left(1-e^{-2r\mu}\right) = \frac{\pi^2}{12\mu}+\frac{1}{2}\log\frac{\mu}{\pi}. \qquad \ldots \qquad \ldots \qquad (34)$$

Again

$$\sum_{r=1}^{\infty} \log\left(1+e^{-r\mu}-e^{-2r\mu}\right) = \sum_{r=0}^{\infty} \log\left(1+e^{-r\mu}-e^{-2r\mu}\right)$$

$$= \int_0^{\infty} \log\left(1+e^{-\mu x}-e^{-2\mu x}\right) dx + \int_0^{\infty} P_1(x) \frac{\mu\left(2e^{-2\mu x}-e^{-\mu x}\right)}{1+e^{-\mu x}-e^{-2\mu x}}\, dx,$$

where

$$P_1(x) = x-[x]-\tfrac{1}{2} = -\sum_{n=1}^{\infty} \frac{\sin n\pi x}{n\pi}.$$

Therefore

$$\sum_{r=1}^{\infty} \log\left(1+e^{-r\mu}-e^{-2r\mu}\right) = \int_0^{\infty} \log\left(1+e^{-\mu x}-e^{-2\mu x}\right) dx - \frac{1}{12}\mu + 0(\mu^3). \quad \ldots \quad (35)$$

Denoting the integral on the right-hand side by I_1 and putting $e^{-\mu x} = \sin^2\theta$, we have

$$I_1 = \frac{2}{\mu}\int_0^{\pi/2} \log\left(1+\sin^2\theta\cos^2\theta\right)\cot\theta\, d\theta$$

$$= \frac{1}{\mu}\int_0^{\pi/2} \frac{\log\left(1+\sin^2\theta\cos^2\theta\right)}{\sin\theta\cos\theta}\, d\theta$$

$$= \frac{1}{\mu}\int_0^{\pi/2} \left\{\sin\theta\cos\theta - \tfrac{1}{2}\sin^3\theta\cos^3\theta + \tfrac{1}{3}\sin^5\theta\cos^5\theta - \ldots\ldots\right\} d\theta$$

$$= \frac{1}{\mu}(\cdot 46313). \qquad \ldots \qquad (36)$$

Substituting (34), (35), (36) in (33) we have

$$\log f(e^{-\mu}) = \frac{\gamma^2}{\mu} + \tfrac{1}{2}\log\left(\frac{\mu}{\pi}\right) + 0(\mu), \qquad \ldots \qquad \ldots \qquad (37)$$

where

$$\gamma^2 = \frac{\pi^2}{12} + \cdot 46313.$$

From (37) we have

$$n = -\frac{\partial}{\partial\mu}\log f(e^{-\mu}) = \frac{\gamma^2}{\mu^2} - \frac{1}{2\mu},$$

which gives

$$\mu \approx \frac{\gamma}{\sqrt{n}} - \frac{1}{4n}.$$

For $z(\mu)$ we get

$$z(\mu) = n\mu + \log f(\mu) = n\left(\frac{\gamma}{\sqrt{n}} - \frac{1}{4n}\right) + \frac{\gamma^2}{4\gamma^2}\left(1 + 4\gamma\sqrt{n}\right) + \tfrac{1}{2}\log\left(\frac{\gamma}{\sqrt{n}} - \frac{1}{4n}\right)$$

$$-\tfrac{1}{2}\log\pi + 0(\mu) \approx 2\sqrt{n}\,\gamma - \tfrac{1}{2}\log\frac{\pi}{\mu}. \qquad (38)$$

Also

$$-\frac{\partial n}{\partial\mu} = \frac{2\gamma^2}{\mu^3} - \frac{1}{2\mu^2}. \qquad (39)$$

Using (38) and (39) in (32) we obtain

$$\omega(n) = \frac{\gamma}{2\pi n}\, e^{2\gamma\sqrt{n}} = \frac{1{\cdot}134}{2\pi n}\exp\left\{2{\cdot}268\sqrt{n}\right\}. \qquad (40)$$

The partitions into integers and half odd integers can be obtained from (40) by replacing n by $2n$ so that

$$P(n) = \frac{1{\cdot}134}{4\pi n}\exp\left\{3{\cdot}206\sqrt{n}\right\}.$$

$P(n)$ thus represents the partitions of n into integers and half odd integers repeated only an odd number of times.

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