DETERMINATION OF ELASTIC CONSTANTS OF SOLIDS BY PULSE METHOD.

By KRISHNAJI, *M.Sc., Department of Physics, University of Allahabad.*

(Communicated by Dr. D. S. Kothari, D.Sc., F.N.I.)

(*Received July 6 ; read August 5, 1949.*)

INTRODUCTION.

One of the most interesting applications of ultrasonics is the determination of elastic constants of solids and determination of the velocity of the two types of elastic waves in them. It is also possible now to study the phenomenon of interference in solid plates and to use the results for determining refractive indices of solids for ultrasonic waves. Some fifty years ago Prof. C. G. Knott (1899) had shown that whenever a wave—say a plane longitudinal one—is incident upon the interface of a liquid and an isotropic solid, it gives rise to three waves, viz. (i) a reflected longitudinal wave in the liquid itself, (ii) a refracted longitudinal wave in the solid, and (iii) a refracted shear wave in the solid; all of them travelling with different velocities. The situation is shown diagrammatically in Fig. 1 where θ is

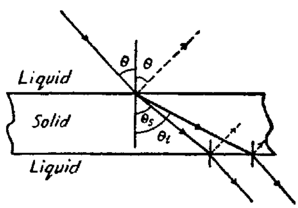

FIG. 1.

the angle of reflection of (i), θ_l is the angle of refraction for (ii) and θ_s is the angle of refraction for (iii). He has developed the theory of propagation of these waves and has further calculated the energies of the reflected and refracted waves for various angles of incidence. A short summary of Knott's (1899) theory applicable to the case of 'longitudinal wave incident from a fluid and falling upon an elastic solid', is appended here for ready reference.

The equations of motion for plane waves in an elastic solid are expressible in the form

$$(m+n)\nabla^2\phi = \rho\frac{d^2\phi}{dt^2} \quad \ldots \quad (1)$$

$$n\nabla^2\psi = \rho\frac{d^2\psi}{dt^2} \quad \ldots \quad (2)$$

$$n\nabla^2\zeta = \rho\frac{d^2\zeta}{dt^2} \quad \ldots \quad (3)$$

where the plane XY is taken perpendicular to the wave front, the displacement in directions X, Y, Z are

$$\xi = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}, \quad \eta = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}, \quad \zeta = \zeta \quad \ldots \quad (4)$$

and ρ is the density, n the rigidity and $m - \frac{1}{3}n = k$ called the bulk modulus.

The stress components are

$$\left.\begin{aligned} P &= (m+n)\nabla^2\phi - 2n\frac{\partial \eta}{dy}, \quad S = n\frac{\partial \zeta}{\partial y} \\ Q &= (m+n)\nabla^2\phi - 2n\frac{\partial \xi}{\partial x}, \quad T = n\frac{\partial \zeta}{\partial x} \\ R &= (m-n)\nabla^2\phi, \quad U = n\left(2\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right) \end{aligned}\right\} \quad \ldots \quad (5)$$

The waves of the ϕ type are longitudinal travelling with the velocity

$$V_l = \sqrt{\frac{m+n}{\rho}} = \sqrt{\frac{E(1-\mu)}{(1+\mu)(1-2\mu)\rho}}. \quad \ldots \quad (6)$$

Where E is the Young's modulus of elasticity and μ is the Poisson's ratio given by the expression

$$\mu = \left[\frac{k^2-2}{2(k^2-1)}\right], \text{ where } k = \frac{V_l}{V_s}, \quad \ldots \quad (7)$$

the waves of the ψ type are shear waves travelling with the velocity

$$V_s = \sqrt{\frac{\eta}{\rho}}. \quad \ldots \quad (8)$$

Let us now consider a specific case of plane waves travelling in a fluid medium and incident upon a solid surface. The surface of separation being defined by the plane $x = 0$. It can be easily seen that (assuming that no ψ waves exist in the liquid; and taking ϕ of the form $Ae^{ib(cx+y+wt)}$) the solution is of the form

$$\phi = Ae^{ib(cx+y+wt)} + A_1e^{ib(-cx+y+wt)}$$

in the liquid medium characterized by the constants m_1, n_1 and ρ_1

$$\phi' = A'e^{ib(c'x+y+wt)}$$

$$\psi' = B'e^{ib(\gamma'x+y+wt)} \quad \ldots \quad (9)$$

refracted waves in the solid medium having constants m_2, n_2 and ρ_2 where A is the amplitude of incident longitudinal wave

A_1 ,, ,, ,, ,, reflected ,, ,,
A' ,, ,, ,, ,, refracted ,, ,,
B' ,, ,, ,, ,, ,, shear ,,
c ,, cotangent θ
c' ,, ,, θ_l
γ' ,, ,, θ_s.

Now the following boundary conditions have to be satisfied at the interface :—

(1) The displacement normal to the surface must be same in both the media.
(2) The tangential displacement must be the same in both the media.
(3) The normal stresses must be equal in both the media.
(4) The tangential stresses must be equal in both the media.

Condition (1) wants

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi'}{\partial x} + \frac{\partial \psi'}{\partial y}$$

which on substitution from (9) gives

$$c(A - A_1) = A'c' + B'$$

or

$$cY = A'c' + B' \quad \ldots \quad (10)$$

where

$$Y = A - A_1.$$

Since it has been assumed that no shear waves exist in liquid, condition (2) need not be considered.
Condition (3) gives in substitution from (5)

$$\frac{\rho_1}{\rho_2}(\gamma'^2 + 1)X = A'(\gamma'^2 - 1) + 2B'\gamma' \quad \ldots \quad (11)$$

where

$$X = A + A_1.$$

Condition (4) gives on substitution from (5)

$$B'(\gamma'^2 - 1) - 2A'c' = 0. \quad \ldots \quad (12)$$

Equations (10), (11), and (12) give on simplification

$$\left.\begin{aligned} Y &= \frac{c'}{c}\,\frac{\gamma'^2 + 1}{\gamma'^2 - 1}\,A' \\ X &= \frac{\rho_2}{\rho_1}\,A'\left[\frac{\gamma'^2 - 1}{\gamma'^2 + 1} + \frac{4c'\gamma'}{\gamma'^4 - 1}\right] \\ B' &= \frac{2c'}{\gamma'^2 - 1}\,A' \end{aligned}\right\} \quad \ldots \quad (13)$$

We can easily see that knowledge of the velocities of the various types of elastic waves can enable us to evaluate the elastic constants of solids by equations (6), (7) and (8) and that amplitudes of the reflected and refracted waves of the two types can be predicted by equations (13). The energy fractions of different waves can be subsequently calculated from the Knott's energy equation

$$\left.\begin{aligned} 1 &= A_1^2 + \frac{\rho_2 c' A'^2}{\rho_1 c} + \frac{\rho_2 \gamma' B'^2}{\rho_1 c} \\ \text{or} \quad 1 &= E_1 + E_2 + E_3 \end{aligned}\right\} \quad \ldots \quad (13)'$$

where E_1, E_2, E_3 denote energies of reflected, refracted longitudinal and refracted shear waves taking the incident energy as unity.

Methods for finding V_l and V_s for opaque solids were first suggested by Bār and Walti (1934) and Bez Bardilli (1935). They have been recently used by Schneider and Burton (1949) for studying the elastic constants of aluminium, copper and several types of resins. The method is based on the phenomenon of total reflection. A longitudinal ultrasonic wave is incident on a plane parallel plate of a solid suspended in a liquid bath. The angle of incidence of the beam is changed by rotating the plate about the vertical axis. The energy transmitted through the plate is noted for increasing angle of incidence. Since the velocity of the waves in the solid is normally greater than in the liquid the waves are refracted away from the normal

(Fig. 1) and therefore at a critical angle of incidence θ_1 the longitudinal wave is totally reflected showing a pronounced minimum in the transmitted energy. Again at another angle θ_2 the shear wave is also totally reflected showing zero transmission of sound energy. At these angles the following well-known relations hold good:

$$V_l = \frac{V}{\sin \theta_1}, \quad \ldots \quad (14)$$

and

$$V_s = \frac{V}{\sin \theta_2}, \quad \ldots \quad (15)$$

where V is the velocity of the longitudinal wave in the liquid. Thus by experimentally finding the values of θ_1 and θ_2 we can know V_l and V_s and using them calculate the values of the elastic constants of the solid.

The possibilities and importance of ultrasonic methods have been greatly enhanced by the use of pulse technique developed largely during the last War. The pulse technique is the latest and most powerful method. The results obtained by this method may be regarded as physically equivalent to those obtained by continuous methods, since it may be shown that differences in behaviour between pulsed sound and continuous sound in liquids should produce negligible effects. Pellam and Galt (1946) have used this technique for measurement of velocity and attenuation of ultrasonic waves in liquids. This paper is divided into two sections; first giving an account of the measurement of the elastic constants of aluminium, brass, iron and a specimen of black stone from Malabar (South India); the second section gives the calculated energies of reflected and refracted waves based on equations (13) and (13)′ for the cases of water-iron and water-malabar stone using the results obtained in the first section. We have, however, postponed the consideration of absorption coefficient and the experimental verification of the results calculated in the second section of this paper for another occasion.

Section I.

Apparatus and method.

Fig. 2 shows the block diagram of the apparatus used. An accurately controlled linear saw-tooth oscillator is used to trigger an electronic radio-frequency pulse generator (1948). The transmitter generates a short radio-frequency pulse of 2·5 mc./sec. repeated at the frequency of the saw-tooth time base oscillator. The

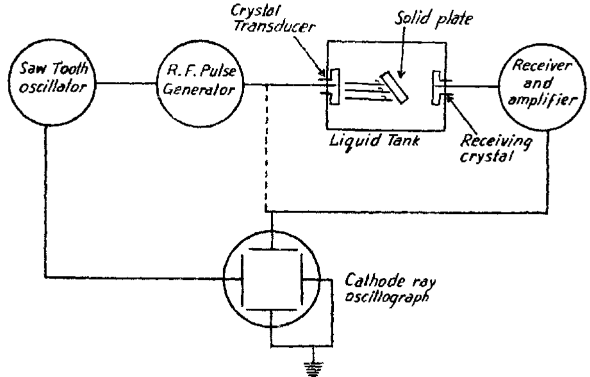

FIG. 2

electrical energy is converted to ultrasonic energy by applying the electrical pulse across a 2·5 mc./sec. X-cut quartz disc contained in a special mount. The transmitter disc is fitted into a specially constructed tank (shown in Fig. 3) which contains the liquid. On the other side of the tank a similar quartz disc is placed which acts as the receiver, i.e., voltage is developed across it when ultrasonic energy falls on it. In the middle of the tank the solid plate is suspended by a specially designed holder. It can be rotated about the vertical axis and the rotation can be measured on the graduated circular dial. Special care is taken to keep the faces of the transmitting and receiving crystal and the solid plate parallel. The output of the receiving crystal is fed to a pulse receiver and amplifier. The equipment is used in conjunction with a cathode ray oscillograph which enables visual or photographic observation. The output of the receiver is applied to the vertical deflection plates of the C.R.O. A portion of the transmitter output is also applied to the same set of plates. To the horizontal plates is applied the saw-tooth time base voltage. When the equipment is switched on we see a pattern as shown in Fig. 4, on the screen. The first vertical pip is due to the transmitter pulse and the second is due to the pulse received by the receiving crystal, the pulse having travelled through the liquid and the solid plate.

The general procedure is to note the height of the second pip or the voltage as the solid plate is rotated, i.e., the angle of incidence is changed. Graphs of amplitude of the pulse passed through the plate and the angle of incidence is plotted for various substances. Photographs of the oscillograph screen is also taken at various angles of incidence.

Observations and Results.

Figures 4, 5 and 6 are photographs showing the pip height for the case of aluminium at zero incidence, when longitudinal wave is totally reflected and when shear wave is totally reflected respectively. Note the complete absence of the second pip in Fig. 6. Figures 7 to 10 are photographs for the case of iron plate at four angles of incidence. Fig. 8 is for the case when longitudinal wave is totally reflected whereas Fig. 10 is for the case when shear wave is also reflected. Figs. 7 and 9 show the interesting phenomenon of multiple reflection between the plate and the receiving electrode. Note the decreasing height of the successive pulses. This will be studied later on and will be utilized to find the attenuation coefficient.

Figures 11 to 16 show the graphs of amplitude of the pulse passed through the plate vs. angle of incidence for various substances. The graphs are self-explanatory. The tank liquid in all the cases was distilled dust-free water.

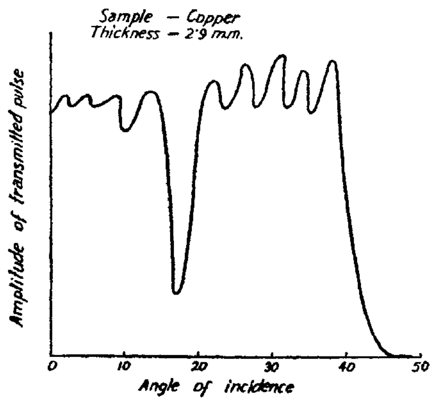

FIG. 11.

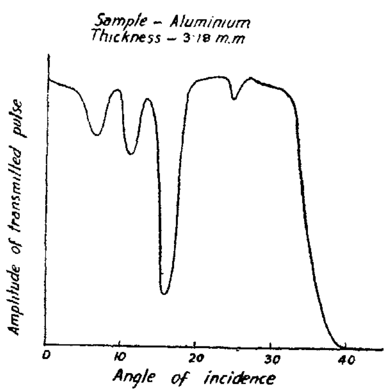

FIG. 12.

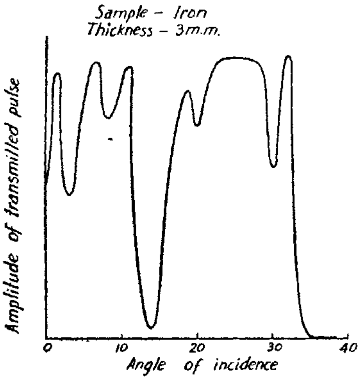

FIG. 13.

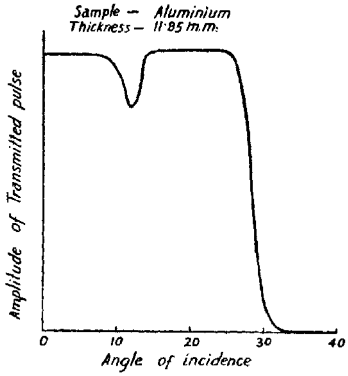

FIG. 14.

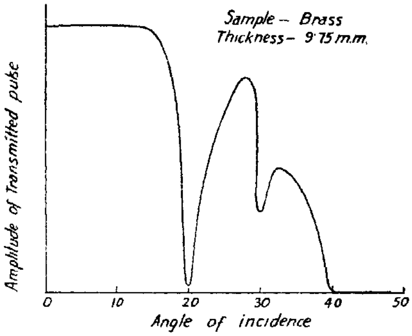

FIG. 15.

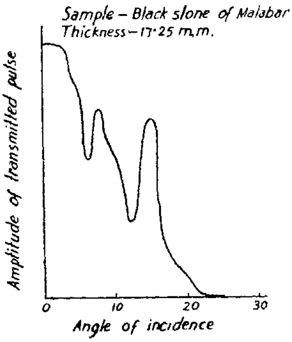

FIG. 16.

It is seen that as expected results obtained from thin plates are low whereas thicker plates (at least a few wavelengths thick) give correct results. The results obtained from these graphs are tabulated in Table I. It is to be noted that in the case of the specimen of black stone from Malabar the shear velocity is larger than in metal.

TABLE I

No.	Substance	Density	Tank Liquid	V cm./sec.	V_l cm./sec.	V_s cm./sec.	η dynes/cm.^{2}	E dynes/cm.^{2}	μ
1	Aluminium	2·70	Water	$1{\cdot}5\times10^5$	$7{\cdot}12\times10^5$	$2{\cdot}83\times10^5$	$2{\cdot}16\times10^{11}$	$6{\cdot}09\times10^{11}$	0·406
2	Brass	8·54	,,	,,	$4{\cdot}5\times10^5$	$2{\cdot}33\times10^5$	$4{\cdot}64\times10^{11}$	$1{\cdot}22\times10^{12}$	0·317
3	Iron	7·86	,,	,,	$6{\cdot}1\times10^5$	$2{\cdot}85\times10^5$	$6{\cdot}38\times10^{11}$	$1{\cdot}78\times10^{12}$	0·361
4	Black stone from Malabar	2·99	,,	,,	$7{\cdot}23\times10^5$	$3{\cdot}84\times10^5$	$4{\cdot}41\times10^{11}$	$1{\cdot}15\times10^{12}$	0·303

SECTION II.

This section gives the calculated values of the energies of the reflected and refracted waves based on equations (13). Table II shows the data for the case of water and iron and the Table III shows the data for the case of water and malabar stone.

TABLE II.
Water to Iron.

θ	E	E_1	θ_l	E_2	θ_s	E_3
0°	1	0·8928	0°	0·1004	0°	0·0000
5°	1	0·8728	20°44′	0·1313	9°32′	0·0059
10°	1	0·8868	44°48′	0·0925	19°15′	0·0207
12°	1	0·8953	57°35′	0·0804	23°17′	0·0245
14°	1	0·9239	79°5′	0·0587	27°22′	0·0124
16°	1	0·8485	....	Does not exist	31°39′	0·1507
18°	1	0·8165	....	,,	35°57′	0·2006
20°	1	0·8550	....	,,	40°32′	0·1410
30°	1	0·8041	....	,,	71°48′	0·1959

TABLE III.
Water to Malabar Stone.

θ	E	E_1	θ_l	E_2	θ_s	E_3
0°	1	0·7586	0°	0·2421	0°	0·0000
5°	1	0·7566	24°51′	0·2180	12°54′	0·0254
10°	1	0·7636	56°47′	0·1514	26°23′	0·0849
12°	1	0·9759	....	Does not exist	32°9′	0·0151
14°	1	0·6451	....	,,	38°15′	0·3549
16°	1	0·6802	....	,,	44°52′	0·3198
18°	1	0·7104	....	,,	52°17′	0·2991
22°	1	0·7989	....	,,	73°32′	0·1986

The experimental verification to these data will be undertaken shortly.

In conclusion I wish to express my sincere thanks to Dr. R. N. Ghosh, D.Sc., F.N.I., F.A.S., under whose careful guidance this work has been conducted.

References.

Bar, R. and Walti, A. (1934). Determination of Poisson's Ratio by Ultrasonic vibrations. *Helv. Phys. Acta*, 7, **7**, 658.

Bez-Bardili, W. (1935). Uber ein Ultraschall-Totalreflektometer zur Messung von Schallgeschwindegk-eiton sowie der elastischen konstanten foster korper. *Zeitschrift FurP hysik.*, **96**, 761.

Glasoe and Lebacqz (1948). Pulse Generators (Radiation Lab. Series McGraw Hill Book Co.).

Knot, C. G. (1899). Reflexion and Refraction of Elastic waves with seismological Applications. *Phil. Mag. and Jour. of Science*, **48**, 64.

Pellam, J. R. and Galt, J. K. (1946). Ultrasonic propagation in Liquids. Application of Pulse Technique to velocity and Absorption Measurements at 15 Megacycles. *Jour. Chemical Physics*, **14**, 608.

Schneider, W. C., and Burton, C. J. (1949). Determination of the Elastic constants of Solids by ultrasonic methods. *Jour. Applied Physics*, **20**, 48.

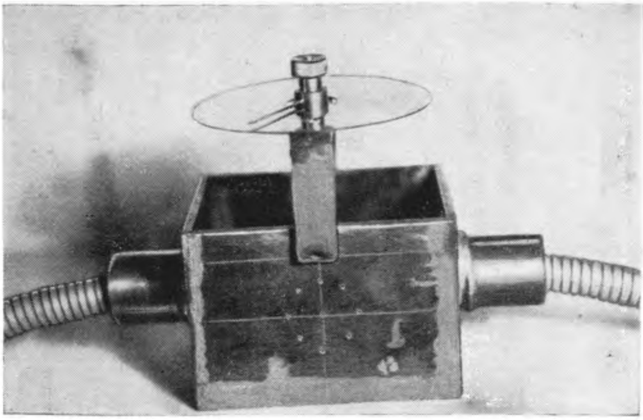

FIG. 3.

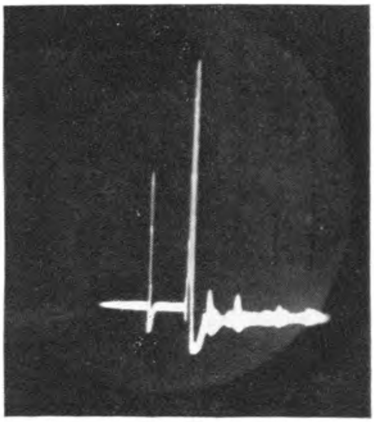

FIG. 4.

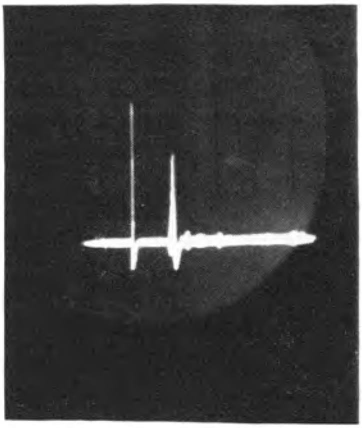

FIG. 5.

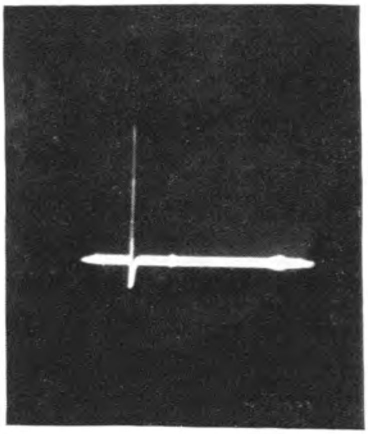

FIG. 6.

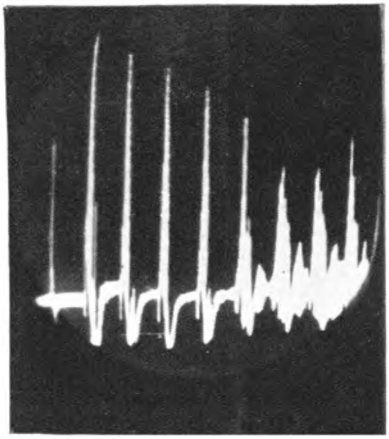

FIG. 7.

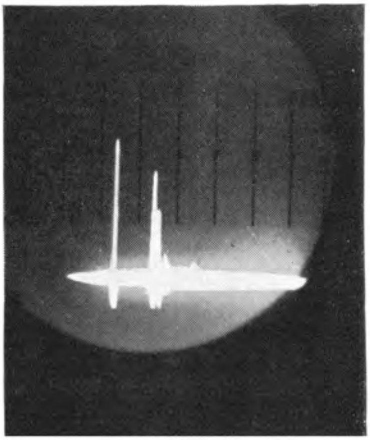

FIG. 8.

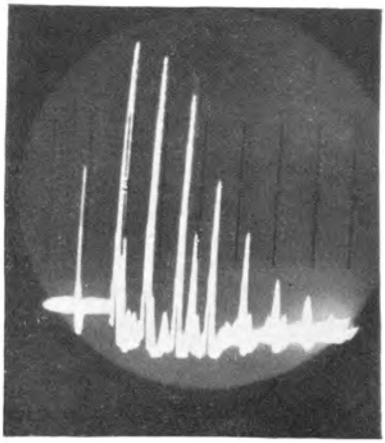

FIG. 9.

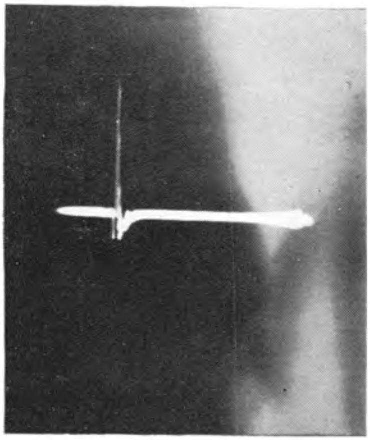

FIG. 10.