THE NONLINEAR METHOD OF CIRCUIT ANALYSIS.*

By Rais Ahmed, *Aligarh University.*

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Summary.

Nonlinear differential equations are being commonly used to express various physical phenomena. Systematic methods of solving such equations have been developed in recent years and they are being applied to problems in physics and engineering. The latest application of nonlinear analysis to vacuum tube circuits has been examined and its limitations discussed.

Nonlinear differential equations are a common phenomenon in the solution of many problems in modern physics and engineering. In the past several years an elaborate method of solving many such equations has been developed both by physicists and mathematicians, and this mathematical tool has been used extensively in the treatment of problems in mechanics, sound and especially electrical engineering. Since the current-voltage relationships in vacuum tubes are nonlinear, an exact analysis of circuits which include vacuum tubes would have to depend on the use of nonlinear differential equations.

One example of the nonlinear differential equation is the famous van der Pol equation for relaxation oscillations:

$$x'' - \mu(1-x^2)x' + x = 0 \quad .. \quad .. \quad .. \quad .. \quad (1)$$

The nonlinearity of this equation consists in the fact that the coefficient of x' depends on the value of x itself, and if μ is positive it is clear that this coefficient, which also represents damping in the circuit, will change in sign as x increases from zero to a value greater than one.

Analytic methods of treatment exist for such equations as the van der Pol's, in the case in which μ is small. Attempts have been made to extend these methods to cases in which μ is large (Minorski, 1947), and indeed to more general equations of the type

$$x'' + f(x, x')x' + g(x) = 0 \quad .. \quad .. \quad .. \quad .. \quad (2)$$

with certain limitations on the functions f and g (Levinson, 1942; Shohat, 1944). However for large μ there is as yet no complete analytic theory comparable to the one that exists for small values of μ.

Since cases with $\mu \gg 1$ are common in unstable circuits (and in many other problems in mechanics), the above-mentioned difficulties led the Soviet physicists L. Mandelstam and N. Papalexi (1935) to develop another approach to this problem (refer to Minorsk, 1947 and Kryloff, 1943). This is the so-called Discontinuous Theory of Relaxation Oscillations, and it has already proved very useful in many branches of physics.

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The Discontinuous Theory of Relaxation Oscillations.

The name of this theory signifies the use of mathematical discontinuities to describe rapidly changing dynamical processes during certain time intervals. This mathematical concept of discontinuities has been used in the classical theory of mechanical impacts where, in the infinitesimally small duration of the impact the dynamics of the process is ignored and the pre- and after-impact conditions are correlated on the basis of such general principles as the conservation of energy or momentum. The method makes it possible to understand correctly the overall effect of the impact, although it precludes a study of the process of the impact itself. The discontinuous theory of relaxation oscillations is based on a similar principle and in order to grasp the phenomenon in the large certain local details are necessarily sacrificed.

In this method relaxation oscillations are defined as those 'quasi-discontinuous oscillations in which the rapid changes between certain levels of a physical quantity occur as the result of the loss of a certain internal equilibrium in the system'. Since a rapidly changing process is being represented by an idealization, namely a mathematical discontinuity, it is obvious that the result will be more realistic if the change is more rapid. To approximate this condition in ordinary circuits one may neglect small oscillatory parameters and thereby conveniently reduce the order of the differential equations involved.

The circuit of Fig. 1 may be considered. Assuming that the left-hand branch has no parasitic capacitance and that the right-hand branch has no parasitic

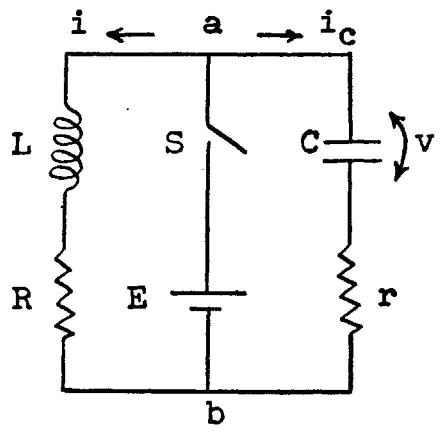

FIG. 1.

inductance and also that the switch S operates instantaneously, the following equations can be written:

$$L\frac{di}{dt}+Ri=E \quad \ldots \quad (3)$$

$$rC\frac{dv}{dt}+v=E \quad \ldots \quad (4)$$

If the voltage across ab is supposed to have been zero before the instant of switching t_0 and equal to a constant E after t_0, the solutions of the above differential equations are:

$$i=\frac{E}{R}(1-\epsilon^{-Rt/L})\,;\ v=E\,(1-\epsilon^{-t/rC}) \quad \ldots \quad (5)$$

which give

$$\frac{di}{dt} = \frac{E}{L}\epsilon^{-Rt/L}; \quad \frac{dv}{dt} = \frac{E}{rC}\epsilon^{-t/rC} \qquad \ldots \quad \ldots \quad \ldots \quad (6)$$

It is seen that for $t = 0$, the solutions $i(t)$ and $v(t)$ are continuous but that their derivatives experience discontinuous jumps. It may be emphasized that $\frac{di}{dt}$ and $\frac{dv}{dt}$ are represented to have discontinuities only because (1) the time interval during which rapid changes occur is considered to be infinitely short, and (2) the parasitic parameters are neglected making it possible to deal with degenerate equations of the first order instead of the full equations of the second order.

With particular reference to degeneration of a second order differential equation

$$ax''+bx'+cx = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

it may be said that on comparison of its solutions (1) when a is small, and (2) when $a = 0$, it is found that although one solution uniformly converges into the other as $a \to 0$, the derivatives similarly converge only for values of t not in the neighbourhood of $t = 0$. In other words near $t = 0$ the derivatives of the two solutions do not match. It is recognized, therefore, that as a result of certain idealizations, like assuming $a = 0$, or $c = 0$ in the differential equation, discontinuities appear in the mathematical treatment of physical phenomena undergoing rapid changes at certain points of their cycles. In view of the above following definition and basic assumption play a very important rôle in the discontinuous theory of relaxation oscillations.

Definition: Critical points are the points at which the differential equation describing a phenomenon in a certain domain ceases to describe it.

Basic Assumption: If the phenomenon is represented by a curve on a particular plane (like the plane of x and x', or the plane of currents at two different points in the circuit) and if a locus of critical points is also drawn on the same plane, then whenever the representative point following the curve of the differential equation describing the phenomenon reaches a critical point a discontinuity occurs in the variable of the system.

To illustrate the meaning suppose the system is described by the differential equations

$$x' = \frac{P(x, y)}{R(x, y)} \text{ and } y' = \frac{Q(x, y)}{R(x, y)} \qquad \ldots \quad \ldots \quad \ldots \quad (8)$$

If (x_c, y_c) is a point such that $R(x_c, y_c) = 0$ then the differential equations become meaningless at that point. Obviously, then $R(x_c, y_c) = 0$ represents the locus of the critical points in this particular case. The curve representing the differential equation itself may be found by eliminating time variations from the above equations. That is,

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

represents such a curve or trajectory.

Mandelstam proposed further the application of the principle of conservation of energy to govern the nature of the 'jump' at the discontinuity. This assumption is physically true since it amounts to assuming continuity of current through an inductance or of voltage across a capacitance in electrical circuits.

With this preliminary discussion a study of the multivibrator of Abraham and Bloch (1927) is presented to illustrate the application of the discontinuous theory to relaxation oscillations. This example is taken from N. Minorski's interpretation of the original by A. Andronov and S. Chaiken. It has only been changed in so far as the plate currents of the tubes are considered functions of both the grid and plate voltages, instead of functions of just the grid voltage, as has apparently been assumed by Andronov and Chaiken. Further, the analysis applies when the grid voltage of neither of the tubes is at or above zero volts.

With reference to Fig. 2 the following equations can be written:

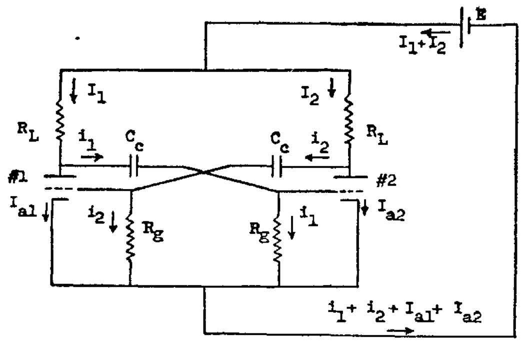

FIG. 2. Symmetrical multivibrator.

$$I_1 = I_{a1}+i_1\,;\; I_2 = I_{a2}+i_2$$

$$R_L I_1+\frac{1}{C_c}\int i_1 dt+R_g i_1 = E\,;\; R_L I_2+\frac{1}{C_c}\int i_2 dt+R_g i_2 = E$$

$$I_{a1} = \theta(e_g, e_p) = \theta(R_g i_2, E-I_1 R_L) \quad \ldots \quad (10)$$

$$I_{a2} = \theta(R_g i_1, E-I_2 R_L)$$

From the equation for the tube currents

$$\frac{dI_{a1}}{dt} = R_g\frac{di_2}{dt}\cdot\frac{\partial\theta}{\partial(R_g i_2)} - R_L\frac{dI_1}{dt}\cdot\frac{\partial\theta}{\partial(E-I_1R_L)} \quad \ldots \quad (11)$$

and

$$\frac{dI_{a2}}{dt} = R_g\frac{di_1}{dt}\cdot\frac{\partial\theta}{\partial(R_g i_1)} - R_L\frac{dI_2}{dt}\cdot\frac{\partial\theta}{\partial(E-I_2R_L)} \quad \ldots \quad (12)$$

The following notation may now be introduced *:

$$\frac{\partial\theta}{\partial(R_g i_2)} = \frac{\partial I_{a1}}{\partial(e_{g1})} = g_{m1}\,;\; \frac{\partial\theta}{\partial(R_g i_1)} = g_{m2}$$

$$\frac{\partial\theta}{\partial(E-I_1R_L)} = \frac{\partial I_{a1}}{\partial(e_p)} = \frac{1}{r_{p1}}\,;\; \frac{\partial\theta}{\partial(E-I_2R_L)} = \frac{1}{r_{p2}}$$

The rate of change of tube currents may be expressed:

$$\frac{dI_{a1}}{dt} = R_g\frac{di_2}{dt}g_{m1} - \frac{R_L}{r_{p1}}\cdot\frac{dI_1}{dt} \quad \ldots \quad (13)$$

$$\frac{dI_{a2}}{dt} = R_g\frac{di_1}{dt}\cdot g_{m2} - \frac{R_L}{r_{p2}}\cdot\frac{dI_2}{dt}\,. \quad \ldots \quad (14)$$

* It is to be noted that these partial derivatives correspond to the tube transconductances and plate resistances and are functions of the grid and plate voltages of the tubes.

If I_1 and I_2 are replaced by $I_{a1}+i_1$ and $I_{a2}+i_2$ respectively then

$$\frac{dI_{a1}}{dt}=\frac{R_g\cdot\frac{di_2}{dt}g_{m1}-\frac{R_L}{r_{p1}}\cdot\frac{di_1}{dt}}{1+\frac{R_L}{r_{p1}}} \quad \ldots \quad (15)$$

and

$$\frac{dI_{a2}}{dt}=\frac{R_g\frac{di_1}{dt}g_{m2}-\frac{R_L}{r_{p2}}\cdot\frac{di_2}{dt}}{1+\frac{R_L}{r_{p2}}} \quad \ldots \quad (16)$$

yielding finally the circuit equations:

$$\left(R_L+R_g-\frac{R_L^2/r_{p1}}{1+R_L/r_{p1}}\right)\frac{di_1}{dt}+\frac{1}{C_c}i_2+\left(R_L+R_g-\frac{R_L^2/r_{p2}}{1+R_L/r_{p2}}\right)=0 \quad \ldots \quad (17)$$

From the above, by algebraic reduction the following equation is obtained:

$$\frac{di_1}{dt}=\frac{\left(R_L+R_g-\frac{R_L^2/r_{p2}}{1+R_L/r_{p2}}\right)\frac{i_1}{C_c}-\frac{R_LR_gg_{m1}i_2}{1+R_L/r_{p1}}\cdot\frac{1}{C_c}}{\frac{R_L^2R_g^2g_{m1}g_{m2}}{(1+R_L/r_{p1})(1+R_L/r_{p2})}-\left(R_L+R_g-\frac{R_L^2/r_{p1}}{1+R_L/r_{p1}}\right)\cdot\left(R_L+R_g-\frac{R_L^2/r_{p2}}{1+R_L/r_{p2}}\right)}$$

$$\equiv\frac{P}{R}. \quad \ldots \quad (18)$$

Note.—Equations (17) and (18) have their counterparts in terms of di_2/dt.

From equation (18) and its counterpart, the equation for the trajectories in the plane of i_1 and i_2 is found,

$$\frac{di_1}{di_2}=\frac{P}{Q} \quad \ldots \quad (19)$$

The methods of nonlinear mechanics can be applied to this equation with the results mentioned below:

(1) The point $i_1=i_2=0$ is a singular point because the function di_1/di_2 becomes indeterminate at this point.

(2) The nature of this singularity may be investigated by noticing that

$$\left(R_L+R_g-\frac{R_L^2/r_p}{1+R_L/r_p}\right)$$

is always positive, and that near $i_1=i_2=0$, the expression $(g_{m1}/(1+R_L/r_{p1})$ may be replaced by a constant k_1, and $(g_{m2}/(1+R_L/r_{p2})$ by another constant k_2, thus making it possible to write:

$$\frac{di_1}{dt}=\left(R_L+R_g-\frac{R_L^2/r_{p2}}{1+R_L/r_{p2}}\right)\frac{i_1}{M}-\frac{R_LR_gk_1}{M}i_2 \quad \ldots \quad (20)$$

and

$$\frac{di_2}{dt}=\left(R_L+R_g-\frac{R_L^2/r_{p1}}{1+R_L/r_{p1}}\right)\frac{i_2}{M}-\frac{R_LR_gk_2}{M}i_1 \quad \ldots \quad (21)$$

where

$$M = C_c \left[R_L{}^2 R_g{}^2 k_1 k_2 - \left(R_L + R_g - \frac{R_L{}^2/r_{p1}}{1+R_L/r_{p1}} \right) \left(R_L + R_g - \frac{R_L{}^2/r_{p2}}{1+R_L/r_{p2}} \right) \right] \neq 0.$$

The characteristic equation of this system then becomes

$$\lambda^2 - \left(2R_L + 2R_g - \frac{R_L{}^2/r_{p1}}{1+R_L/r_{p1}} - \frac{R_L{}^2/r_{p2}}{1+R_L/r_{p2}} \right) \frac{\lambda}{M} - \frac{1}{C_c M} = 0 \quad \ldots \quad (22)$$

The nature of the roots of this equation obviously depends on the sign of M. When M is negative, the origin is a stable point, and hence this condition is not of interest for relaxation oscillations. When M is positive, then both the roots of (22) are real and bear opposite signs, signifying that the point $i_1 = i_2 = 0$ is unstable. It is further to be noted that as i_1 and i_2 increase (one positively and the other negatively) the factor k involved in the expression for M, changes. This change will make one of the k's decrease to zero, making M negative and allowing the circuit to relax.

(3) According to the theorem of I. Bendixson the system can possess no closed analytic trajectories if the expression

$$\frac{\partial P}{\partial i_1} + \frac{\partial Q}{\partial i_2},$$

referring to the differential equation (18) and its counterpart does not change sign in a certain domain of i_1 and i_2. The sum of the two partial derivatives in the present case will be

$$\left(2R_L + 2R_g - \frac{R_L{}^2/r_{p1}}{1+R_L/r_{p1}} - \frac{R_L{}^2/r_{p2}}{1+R_L/r_{p2}} \right) \Big/ C_c$$

which is permanently positive. No closed trajectories are, therefore, possible.

(4) The equation for the critical points is the following:—

$$R_L{}^2 R_g{}^2 k_1 k_2 - \left(R_L + R_g - \frac{R_L{}^2/r_{p1}}{1+R_L/r_{p1}} \right) \left(R_L + R_g - \frac{R_L{}^2/r_{p2}}{1+R_L/r_{p2}} \right) = 0 \quad \ldots \quad (23)$$

This is roughly sketched in Fig. 3, and as explained before the trajectories given by (19) will approach this locus of the critical points. When the representative point

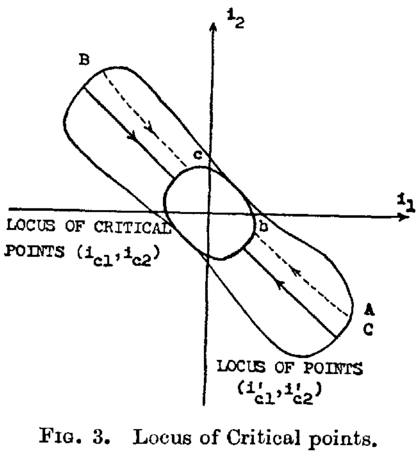

FIG. 3. Locus of Critical points.

moving along a trajectory will reach this locus, it will be shifted instantaneously to some other point from where a new trajectory will be followed. Thus to each point (i_{1c}, i_{2c}) on the locus of the critical points there will be a corresponding point (i'_{1c}, i'_{2c}) such that a new trajectory will start from the latter point. By application of the Mandelstam conditions on the voltage across the condensers, that is by equating voltages at (i_{1c}, i_{2c}) and (i'_{1c}, i'_{2c}), the form of the locus of the points from which the trajectories start can be found. This is also shown in Fig. 3. A trajectory starting at the point A will reach b which is a critical point; a jump bB will then take place with a new trajectory Bc starting from B. From c the representative point will jump to C and a new trajectory will start from there. Ultimately the system will reach a steady state trajectory shown in full lines in Fig. 3.

Discussion of the Discontinuous Theory.

This theory is very effective as long as the assumptions made closely describe the conditions in a particular system. But like all other theories the reliability of the results obtained with the help of this theory is directly proportional to the reality expressed in the assumptions. In the multivibrator operation the time taken for one tube to switch on and for the other to switch off is really negligible when the repetition rate (or the frequency) is small, there is no appreciable effect caused by the inductance in the circuit, and the treatment outlined above very nearly describes the functioning of the circuit. However, if interest is centered on higher frequencies, the time of switch-over assumes greater importance relative to the period between two switch-overs, and the Discontinuous Theory, by its very nature has no information to give on this phenomenon.

At higher frequencies the shunting and interelectrode capacitances, as is well known, become quite important, and when they have to be taken into account the Bendixson criterion, which is only a negative criterion, breaks down. It is evident, therefore, that when the shunting capacitance C is present even the information previously given by the discontinuous treatment (not to say of the information precluded by the method) becomes seriously limited. If the above fact is taken in conjunction with the possibility of the presence of other parasitic inductances and capacitances, it becomes quite evident that the Discontinuous Theory, in spite of its mathematical soundness, becomes a rather complicated tool at the higher frequencies so common in present physics and engineering practice.

In spite of the many developments in the nonlinear method it appears that in order to study circuit behaviour at high frequencies so as to have some quantitative information on transient and fast phenomena, one has to look for some other means of analysis. In doing this some approximations are bound to be made but as long as the task of analysing the circuit is simplified without spoiling the validity of the results for practical application, such approximations can be acceptable. For example, if the operational range of the currents and voltages in the tube is very small it is quite possible to approximate the nonlinear characteristic by a straight line within that range. This limitation simplifies the analysis considerably and makes it possible to write circuit equations in a linear form which then can be solved by well-known methods.

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