ON THE CONSTRUCTION OF MODELS WITH A DISCONTINUITY OF THE MOLECULAR WEIGHT FOR STARS WITH GIVEN VALUES OF THEIR MASS, RADIUS AND LUMINOSITY.

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1. INTRODUCTION.

In recent years considerable attention has been directed to the study of stellar models with a discontinuity of the molecular weight μ somewhere in the stellar material. Schönberg and Chandrasekhar (1942) considered a stellar model with an isothermal core surrounded by an envelope in radiative equilibrium, there being a prescribed discontinuity of the molecular weight across the common interface. Hoyle and Lyttleton (1942, *b*) sought to give a tentative explanation for the structure of red giants by considering the effect on a stellar configuration of a change in molecular weight in the outer region and shewed that under proper restrictions such difference in the atmospheric composition from that of the major part of the mass would cause the star assume a very extensive radius and yet retain its compatibility with the necessary requirements of thermodynamic and mechanical stability. These authors also suggested a mechanism by which the contemplated non-uniformity in the atmospheric composition can be brought about. More recently Ledoux (1947) has examined the effects of a discontinuity in molecular weight occurring in the radiative envelope as well as in the deep interior of a star. He has shewn that in both cases a sharp discontinuity is in general smoothed out and the two regions become separated by a transition zone of variable molecular weight, though a jump in μ of a definite amount is permissible at the outer end of the transition zone. The hypothesis of a sharp discontinuity is indeed a mathematical abstraction, nevertheless the study of solutions of the stellar equations with a sharp discontinuity of μ and satisfying the conditions of stability on both sides of the interface should be considered important from the physical point of view. We assume that such solutions may be associated with configurations physically realisable except for some matters of details.

In the present paper it is proposed to study the problem of the internal constitution of stars on the basis of the convective-radiative model with a sharp discontinuity of composition across the interface. It is easy to see how this difference in composition between the core and the envelope would arise as a consequence of the conversion of hydrogen into helium within the core by the thermonuclear processes responsible for stellar energy generation. The method developed here shews that the equations governing the structure of the star taken along with Bethe's law of energy generation and some very plausible hypothesis regarding the proportion of heavy elements in the stellar material within and outside the convective core, are *just sufficient* to determine the chemical composition of the star from a knowledge of its three observable parameters, viz., its mass (M), radius (R) and luminosity (L). It must not, however, be understood that these equations would always furnish significant solutions for any given L, M, R; in fact, extremely arduous numerical calculations will be required to decide the point in each individual case. We have attempted to work out in this paper the case of the star α Cen A, but have not succeeded in obtaining a significant solution. Other stars might probably furnish admissible solutions, but in view of the complexities involved in each calculation a second attempt has not been made.

2. The equations of equilibrium.

The investigations in the present paper are based on the following assumptions:

(i) The star is a point source model with a convective core surrounded by a radiative envelope.

(ii) The perfect gas law is valid throughout and radiation pressure is negligible.

(iii) The chemical composition (and consequently the mean molecular weight μ) is constant in the core and also in the envelope, but there is a sharp discontinuity of μ across the common interface.

(iv) Photoelectric absorption is the only source of opacity and the guillotine factor in Kramers' formula may be regarded, with reasonable accuracy, as proportional to a power of the density alone (a proper choice of this power will ensure a fair agreement with Morse's table).

The equilibrium equations for the envelope region, embodying the above assumptions, can be written down in the usual notations as

$$\frac{dP}{dr} = \frac{k}{\mu H}\frac{d}{dr}(\rho T) = -G\frac{M(r)}{r^2}\rho \qquad (1)$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho \qquad (2)$$

$$\frac{d}{dr}\left(\tfrac{1}{3}aT^4\right) = -\frac{\kappa\rho}{c}\frac{L}{4\pi r^2}. \qquad (3)$$

Introducing the following representation of the guillotine factor τ (Schwarzschild, 1946),

$$\log_{10}\tau = 0{\cdot}6+\log_{10}\rho^{0\cdot25} \qquad (4)$$

in Kramers' opacity formula, one gets

$$\kappa = \kappa_0\rho^{0\cdot75}T^{-3\cdot5} \quad \text{with } \kappa_0 = 10^{25}(1+X)(1-X-Y), \qquad (5)$$

where X and Y are respectively the hydrogen and helium contents in the stellar material.

The structure of the convective core is governed by the Lane-Emden function of index $n = 3/2$, and the energy production within it according to Bethe's formula gives

$$L = \int_0^{\text{core}} 4\pi r^2\rho\epsilon\, dr \qquad (6)$$

where

$$\epsilon = \epsilon_0 X\rho T^{\eta}. \qquad (7)$$

For sunlike stars, $\eta = 17{\cdot}25$ and $\epsilon_0 = 4{\cdot}085\times10^{-126}$ give a very satisfactory representation of the exact exponential law of energy generation.

3. Integration of the envelope equations.

For purposes of integration it is very convenient to rewrite the envelope equations by introducing the dimensionless variables, p, t, q and x (Schwarzschild, 1946) defined by

$$P = p\frac{GM^2}{4\pi R^4},\ T = t\frac{\mu H}{k}\frac{GM}{R},\ M(r) = qM,\ r = xR. \qquad (8)$$

Equations (1)—(3) then become

$$\frac{dp}{dx} = -\frac{p}{t}\frac{q}{x^2} \quad \ldots \quad (9)$$

$$\frac{dq}{dx} = \frac{p}{t}x^2 \quad \ldots \quad (10)$$

$$\frac{dt}{dx} = -\frac{C}{x^2}\frac{p^{1\cdot75}}{t^{8\cdot25}} \quad \ldots \quad (11)$$

where

$$C = \frac{3\kappa_0}{4ac}\left(\frac{k}{\mu HG}\right)^{7\cdot5}\left(\frac{1}{4\pi}\right)^{2\cdot75}\frac{LR^{1\cdot25}}{M^{5\cdot75}}. \quad \ldots \quad (12)$$

Using again $u = \frac{1}{x} - 1$, one obtains

$$\frac{dp}{du} = \frac{p}{t}q \quad \ldots \quad (13)$$

$$\frac{dq}{du} = -\frac{p}{t(1+u)^4} \quad \ldots \quad (14)$$

$$\frac{dt}{du} = C\frac{p^{1\cdot75}}{t^{8\cdot25}} \quad \ldots \quad (15)$$

as the differential equations to be integrated from the boundary inwards, starting with the initial conditions

$$p = t = 0,\ q = 1,\ \text{when } u = 0. \quad \ldots \quad (16)$$

These equations will furnish a single parametric set of solutions depending on the value of C chosen.

The integrations may be started by constructing approximate analytic solutions for the variables and can be continued up to a certain depth where the approximation q = constant ($=1$) can still be regarded as valid. The procedure is indicated below.

From (13) and (15) it follows

$$\int_0^t t^{7\cdot25}\,dt = \frac{C}{q}\int_0^p p^{0\cdot75}\,dp$$

or

$$\frac{p^{1\cdot75}}{t^{8\cdot25}} = \frac{7}{33}\frac{q}{C}. \quad \ldots \quad (17)$$

Now using (17) in (15), one gets after integration

$$t = \frac{7}{33}qu. \quad \ldots \quad (18)$$

Equation (17) therefore yields

$$p = C^{-4/7}\left(\frac{7}{33}\right)^{37/7} q^{37/7}u^{33/7}. \quad \ldots \quad (19)$$

With these values of p and t, equation (14) integrates out to

$$-\int_1^q q^{-30/7}\,dq = C^{-4/7}\left(\frac{7}{33}\right)^{30/7}\int_0^u u^{26/7}\,(1+u)^{-4}\,du$$

or
$$q^{-23/7} = 1+C^{-4/7}\left(\frac{7}{33}\right)^{30/7}\frac{23}{33}\,u^{33/7}\left(1-\frac{33}{10}u+\frac{330}{47}u^2-\frac{660}{54}u^3+\ldots\right). \tag{20}$$

These expressions can in turn be substituted in the differential equations (13)—(15) and higher approximate solutions derived. The integrations can then be proceeded with by standard numerical methods.

4. Structure of the convective core and conditions at its boundary.

It is well known (Chandrasekhar, 1939) that pressure, temperature and mass distribution within an Emden polytrope of index 3/2 are given by

$$P(\xi) = \frac{k}{\mu H}\rho_c T_c \theta^{2\cdot 5} \quad \ldots \quad (21)$$

$$T(\xi) = T_c\theta \quad \ldots \quad (22)$$

$$M(\xi) = 4\pi\rho_c\left(\frac{5k}{8\pi\mu GH}\frac{T_c}{\rho_c}\right)^{3/2}\left(-\xi^2\frac{d\theta}{d\xi}\right), \quad \ldots \quad (23)$$

where ρ_c, T_c denote the central density and temperature of the configuration respectively. Further the usual radial distance r is connected with the Emden variable ξ by the relation

$$r = \left(\frac{5k}{8\pi\mu GH}\frac{T_c}{\rho_c}\right)^{1/2}\xi\,. \quad \ldots \quad (24)$$

The conditions to be satisfied at the interface between the convective core and the radiative envelope are that the pressure, temperature and mass should be continuous across it. The interface being a surface of discontinuity of μ, the first two conditions would also imply the continuity of ρ/μ. Mathematical expressions for these conditions in terms of the variables for the two regions are

$$p_e\frac{GM^2}{4\pi R^4} = \frac{k}{\mu_i H}\rho_c T_c \theta_i^{2\cdot 5} \quad \ldots \quad (25)$$

$$t_e\frac{\mu_e H}{k}\frac{GM}{R} = T_c\theta_i \quad \ldots \quad (26)$$

$$q_e M = 4\pi\rho_c\left(\frac{5k}{8\pi\mu_i GH}\frac{T_c}{\rho_c}\right)^{3/2}\left(-\xi^2\frac{d\theta}{d\xi}\right)_i \quad \ldots \quad (27)$$

where the indices e and i refer respectively to the envelope and core side values on the common boundary.

The following relation fixing the position of the interface will also be available.

$$x_e R = \left(\frac{5k}{8\pi\mu_i GH}\frac{T_c}{\rho_c}\right)^{1/2}\xi_i\,. \quad \ldots \quad (28)$$

Equation (6) determining the total energy output inside the core now takes the form

$$L = 4\pi\epsilon_0 X_i \rho_c^2 T_c^\eta \left(\frac{5k}{8\pi\mu_i GH}\frac{T_c}{\rho_c}\right)^{3/2} \int_0^{\xi_i} \theta^{\eta+3}\xi^2\, d\xi. \qquad (29)$$

On the hypothesis that the discontinuity in μ arises on account of the conversion of hydrogen into helium (within the convection zone) it is reasonable to assume the content of heavy elements (apart from hydrogen and helium) $1-X-Y$ as even throughout the whole star. This requires

$$1-X_i-Y_i = 1-X_e-Y_e. \qquad (30)$$

Lastly, there is the equation giving the behaviour of the effective polytropic index n, defined by $n+1 = \frac{d \log P}{d \log T}$ on both sides of the interface. Considerations of the sort given by Hoyle and Lyttleton (1946) applied to the present case with the opacity formula (5) lead to the following condition

$$\mu_i^{0\cdot75}(1+X_i)(1-X_i-Y_i)(n_i+1) = \mu_e^{0\cdot75}(1+X_e)(1-X_e-Y_e)(n_e+1). \qquad (31)$$

Now $\qquad n_i+1 = 2{\cdot}5$, for a 3/2 polytrope

and $\qquad n_e+1 = \left(\frac{d \log p}{d \log t}\right)_e = \frac{1}{C}\left(\frac{qt^{8\cdot25}}{p^{1\cdot75}}\right)_e$, for the envelope solution,

so that, equation (31) gives, in view of (30),

$$C = \frac{2}{5}\left(\frac{\mu_e}{\mu_i}\right)^{0\cdot75}\frac{1+X_e}{1+X_i}\left(\frac{qt^{8\cdot25}}{p^{1\cdot75}}\right)_e,$$

whence using (12) one finally obtains

$$\frac{3}{4}\cdot 10^{25}\left(\frac{k}{HG}\right)^{7\cdot5}\frac{1}{ac}\left(\frac{1}{4\pi}\right)^{2\cdot75}\frac{LR^{1\cdot25}}{M^{5\cdot75}}(1+X_e)(1-X_e-Y_e)\times$$

$$\left[2X_e+\frac{3}{4}Y_e+\frac{1}{2}(1-X_e-Y_e)\right]^{7\cdot5} = \frac{2}{5}\left(\frac{1+3X_i+0{\cdot}5Y_i}{1+3X_e+0{\cdot}5Y_e}\right)^{0\cdot75}\frac{1+X_e}{1+X_i}\left(\frac{qt^{8\cdot25}}{p^{1\cdot75}}\right)_e. \qquad (32)$$

It may be noted that in writing equation (32) the formula

$$\mu = \frac{2}{1+3X+0{\cdot}5Y} \qquad (33)$$

has been taken account of.

Equations (25), (26), (27), (28), (29), (30) and (32) must have to be satisfied in order that a model of the type envisaged may exist for a star with definite values of L, M and R. The model will be completely determined if these seven equations can be solved for the seven quantities T_c, ρ_c, ξ_i, X_i, Y_i, X_e and Y_e; L, M and R being assumed given. The problem appears to be just determinate though significant solutions may not always exist. It may be remarked here that for a successful solution of the problem an integration of the envelope equations for a suitable value of the parameter C must be provided. The procedure to be adopted for handling these equations will be discussed later.

5. Stability of the model.

For the present, if the assumption is made that a model has been constructed in conformity with the above requirements, the question of its stability still remains open. We propose to examine this point now.

We start with the observation that a necessary and sufficient condition for the stability of the radiative gradient in a medium with a uniform composition and with a given distribution of density and temperature is the following

$$\left(\frac{d\log P}{d\log T}\right)_{\text{rad}} > \left(\frac{d\log P}{d\log T}\right)_{\text{ad}}. \quad \ldots \quad \ldots \quad \ldots \quad (34)$$

For stability of the adiabatic gradient, the inequality is reversed. It is known that along any solution of the equations of radiative equilibrium with proper boundary conditions, the effective polytropic index n decreases inwards from the outer boundary of the star, so that, if in any case the convective core begins at a point for which n (radiative) exceeds 1·5 (the corresponding adiabatic index), condition (34) is satisfied and stability of the radiative gradient in the entire envelope region is ensured. A little reflection will shew that in the case of a discontinuity of μ as under consideration here, the two gradients cannot merge and have a common value on the surface of the core. In equation (31) we have set $n_i = 1{\cdot}5$, which only means an equality of the radiative and adiabatic gradients on the inner side of the interface. If now $\left(\frac{d\log P}{d\log T}\right)_{\text{rad}} < 2{\cdot}5$ everywhere within the core, we have a stable convective core surrounded by a stable radiative envelope, there being a discontinuity of μ across the interface. The problem therefore is to investigate under what conditions $\left(\frac{d\log P}{d\log T}\right)_{\text{rad}}$, starting with a value 2·5 on the inner boundary of the core, remains less than this value at all interior points. To calculate $\left(\frac{d\log P}{d\log T}\right)_{\text{rad}}$ within the core, recourse must be had to equation (1), and equation (3) with $L(r)$ in place of L on the right hand side. These equations give

$$n+1 = \left(\frac{d\log P}{d\log T}\right)_{\text{rad}} = \frac{16\pi acG}{3\kappa_0}\,\frac{T^{7\cdot 5}}{P\rho^{0\cdot 75}}\,\frac{M(r)}{L(r)} \quad \ldots \quad \ldots \quad (35)$$

whence one obtains, using the perfect gas law and the polytropic equation of state

$$\frac{\delta n}{n+1} = \frac{\delta M(r)}{M(r)} - \frac{\delta L(r)}{L(r)} + \frac{31}{8}\frac{\delta T}{T}. \quad \ldots \quad \ldots \quad \ldots \quad (36)$$

For an outward step $\delta T < 0$ and $\frac{\delta M(r)}{M(r)} > \frac{\delta L(r)}{L(r)}$ (Hoyle and Lyttleton, 1942*a*). Hence n will continue to increase outwards so long as

$$\left(\frac{\delta M(r)}{M(r)} - \frac{\delta L(r)}{L(r)}\right) > \frac{31}{8}\left|\frac{\delta T}{T}\right|. \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

In terms of Emden variables θ, ξ, this inequality becomes

$$\left(\frac{\theta^{3/2}}{(-d\theta/d\xi)} - \frac{1}{I(\xi)}\frac{dI(\xi)}{d\xi}\right) > \frac{31}{8}\left|\frac{1}{\theta}\frac{d\theta}{d\xi}\right| \quad \ldots \quad \ldots \quad \ldots \quad (38)$$

where

$$I(\xi) = \int_0^{\xi} \theta^{\eta+3}\,\xi^2\,d\xi.$$

Denoting the left hand side of (38) by a function $f_1(\xi)$ and the right hand side by $f_2(\xi)$, and taking $\eta = 17{\cdot}25$ one obtains the following table for these functions.

ξ	f_1	f_2
0	0	0
0·4	1·415	0·518
0·6	1·908	0·780
0·8	2·185	1·045
1·0	2·197	1·317
1·2	1·999	1·597
1·4	1·703	1·890
1·6	1·409	2·201

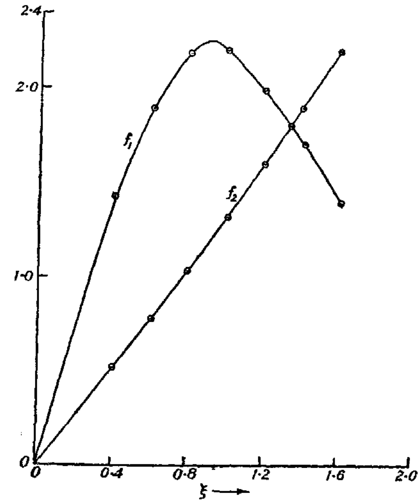

Fig. 1. The functions f_1, f_2, are plotted against ξ.

It will be evident from a plot of these functions as in Fig. 1 that condition (37) is violated as soon as ξ exceeds the value 1·34. This proves that for any convective core of radius $\xi_i \leqslant 1{\cdot}34$, n (starting with a value 1·5 on the inner boundary) will monotonically decrease inwards just in keeping with the requirement of convective stability. This sets an upper limit to the size of the stable convective core for the model we are seeking to build. It is further evident that no stable convective core can begin at an interface $\xi_i > 1{\cdot}34$. This knowledge is important for the trial and error method we shall adopt below for the solution of our problem.

6. Solution of the equations of conditions on the interface.

We shall now discuss the equations (25), (26), (27), (28), (29), (30) and (32), admissible solutions of which, if any, will completely determine the structure of the star under consideration. The method of solution is one of trial and error. A value of the parameter C in the envelope equations is first selected and the corresponding integrations performed. From the given value of L, the luminosity of the star, an approximate value of the central temperature T_c is guessed. This guess will not be wide off the mark, since the energy generation as controlled by Bethe's law is highly sensitive to T_c. A trial value of the central density ρ_c is now taken. In other words, an attempt is made to solve the equations of fit for a given value of C. Our assumed values of T_c and ρ_c are purely trial values.

Equations (27), (28) yield

$$\left(\frac{q}{x^3}\right)_e \frac{M}{R^3} = 4\pi\rho_c\left(-\frac{1}{\xi}\frac{d\theta}{d\xi}\right)_i . \qquad (39)$$

If now the core be supposed to begin at some distance $(x)_e$ along the envelope solution, equation (39) will determine $\left(\frac{1}{\xi}\frac{d\theta}{d\xi}\right)_i$ whence θ, ξ will be obtained by interpolation in tables of Emden functions for $n = 1{\cdot}5$. A value of $(x)_e$ should be fixed by trial so as to give ξ_i within the range $1{\cdot}0 \leqslant \xi_i \leqslant 1{\cdot}34$. A value of $\xi_i > 1{\cdot}34$ is not permissible by stability considerations, and a value $< 1{\cdot}0$ should also be left out in view of the fact that the energy integral in equation (29) does not become sensibly constant at this distance. The range of possible values for ξ_i is thus considerably narrowed down by these requirements. Equations (25) and (26) now give μ_i and μ_e which should however satisfy the condition that $\mu_i > \mu_e$ on account of the probable presence of more helium inside the core than outside.* This value of μ_i substituted in equation (29) would then fix X_i. Equation (33), taken with the indices i and e, and equation (30) would together determine X_e, Y_i and Y_e. These quantities should all lie in the range (0, 1) and also $X+Y$ should be less than unity. Lastly, equation (32) should be used as a check for the fit. A violation of this equation would require the whole calculation to be repeated with a different value of $(x)_e$, keeping the guessed values of T_c and ρ_c unaltered. The possibility of a fit by trial with different values of $(x)_e$ (only those values of $(x)_e$ are admissible for which $n_e > 1{\cdot}5$) should thus be exhausted first. Attempt should then be made to obtain a fit on the same envelope solution with different values of ρ_c. A little practice would enable one to find out the limitations on the possible values of ρ_c. Alterations in the values of T_c should next be attempted. Possible range of variations in T_c will indeed be very small for reasons stated earlier. Finally, the whole sequence of calculations should be repeated with different envelope solutions (for different values of C). Here also by properly taking into account the various causes of failures of the solutions, one can without much difficulty find out the restrictions on the values of C. If any significant solution be still not obtained, the conclusion will be that the equations of fit do not admit a solution for the given values of L, M and R.

In an attempt to construct a model of the type proposed, we have worked out the case of the star α Cen A for which $L = 1{\cdot}26$, $M = 1{\cdot}10$ and $R = 1{\cdot}23$, all in solar units. In a previous paper (Sen and Burman, 1948) on the study of the chemical composition (assumed uniform throughout) of stars on the basis of the Cowling model and Bethe's formula this star was found to present an instance where the equations determining the composition did not offer any significant solutions. We have verified that it is not also possible to determine its composition (supposed uniform) on the basis of Schwarzschild's (1946) model for the sun with the energy generation formula in the form given by Bethe which we have used. For the present work we have constructed numerical solutions of the envelope equations for values of the parameter $C = (2{\cdot}2427,\ 2{\cdot}6427,\ 3{\cdot}4427,\ 3{\cdot}8427)\times 10^{-6}$. Another integration of the equations for $C = 3{\cdot}0427\times 10^{-6}$, provided by Schwarzschild (1946) is also at our disposal. Calculations have been made to obtain solutions of the equations of fit for each of these envelope solutions with values of central temperature in the range $19{\cdot}5 < T_c < 21{\cdot}5$ million degrees, and associated central densities in the range $30 \leqslant \rho_c \leqslant 100$ gm./cm^3. Failure in each case arises at the ultimate stage of the calculation on account of the violation of equation (32) which has been used as a check. In constructing the envelope solutions, values of the parameter have been so selected as to cover the range, somewhat above and below Schwarzschild's value. We may group the failures of our trial solutions into two classes. Firstly, there are trial solutions which prove to be incorrect even before our check equation (32) is arrived at, on account of the failure of some necessary conditions for the correct solution as stated above. There is a second class of trial solutions which work out satisfactorily up to the ultimate stage but only fail at the check equation. If we arrange all trial solutions according to the values of the parameter C in the above range, it is found that the solutions of the second class form a series which is bordered on the two sides by trial solutions of the first class.

* Because of production of helium within at the expense of hydrogen.

The range of C thus examined appears to be sufficient and it is permissible to conclude that no proper solution satisfying all the conditions of the problem exists in the case under consideration.

Lastly there remains the pleasant duty of acknowledging grateful thanks to Professor N. R. Sen for many interesting discussions and to the National Institute of Sciences of India, for the award of a Research Fellowship which enabled the author to carry on the work.

SUMMARY.

The problem of the determination of the chemical composition of a star from a knowledge of its mass, radius and luminosity, is studied from the point of view of Bethe's energy generation law, and on the basis of a convective-radiative model with a sharp discontinuity of composition (and hence of molecular weight) across the interface. It is found that just as many equations governing the structure of the star can be framed as there are unknown quantities required for a complete specification of the chemical composition both inside and outside the convective core. Stability consideration sets an upper limit to the size of the convective core for the correct model. It should be noted, however, that the equations referred to above, should not always furnish a significant solution compatible with all the conditions of the problem. In fact, the case of the star α Cen A worked out in this paper leads to no admissible solution.

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