RAMANUJAN'S FUNCTION WITH RESPECT TO THE MODULUS 49.

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The function $\tau(n)$ generated by

$$\sum_{n=1}^{\infty} \tau(n)x^n = x \prod_{n=1}^{\infty} (1-x^n)^{24}$$

has a number of remarkable properties many of which were first discovered by Ramanujan. In recent years much attention has been paid to congruence properties of the function $\tau(n)$. Bambah, Banerjee, Chowla, Gupta, Lahiri, Ramanathan, Wilton and others have obtained congruence relations with respect to moduli 2^{10}, 3^5, 5^3, 7, 23 and 691. About two years ago the writer noticed, by a study of the values in Watson's table of $\tau(n)$ for $n \leqslant 1000$ a curious property of $\tau(n)$ with respect to the modulus 49, namely: If p and q are two primes differing by a multiple of 49, then $\tau(p)-\tau(q)$ is also divisible by 49, provided p and q are non-residues of 7. This property, with its unusual proviso, continued to hold without exception when Watson's table was extended to $n = 2500$, but defied attempts at proof. Finally a modification of the methods used previously was hit upon which not only yielded a proof of this property, but also results for the moduli 2^{11} and 3^6, which will be discussed in another paper. The present note gives a short proof of the 49 property as expressed by the following theorem.

Theorem. *Let p be a prime congruent to 3, 5, or 6 modulo 7. Then*

$$\tau(p) \equiv 3p(p^3+1) \qquad (\text{mod } 49).$$

It may be shown by referring to tables of $\tau(n)$ that the restriction of p to the abovementioned residue classes modulo 7 is a necessary one and that $\tau(p)$, for p an unrestricted prime, is not congruent to a polynomial in p (mod 49).

The proof of the Theorem may be considerably shortened by referring to the following definitions and identities of Ramanujan.

$$\Phi_{r,s} \equiv \Phi_{r,s}(x) = \sum_{n=1}^{\infty} n^r \sigma_{s-r}(n) x^n$$

$$P = 1-24\Phi_{0,1}$$

$$Q = 1+240\Phi_{0,3}$$

(1) $$R = 1-504\Phi_{0,5}$$

(2) $$\Delta = 1728 \sum_{n=1}^{\infty} \tau(n)x^n = Q^3-R^2$$

(3) $$720\Phi_{1,4} = PQ-R$$

(4) $$1584\Phi_{1,10} = 3Q^3+2R^2-5PQR = 3\Delta-5R(PQ-R)$$

$$(5)\qquad 41472\,\Phi_{4,7} = 3Q^3+4R^2+7(P^4Q-4P^3R+6P^2Q^2-4PQR)$$
$$= 3\Delta+7(P^4Q-4P^3R+6P^2Q^2-4PQR+R^2)$$

$$(6)\qquad 3617+16320\,\Phi_{0,15} = 1617Q^4+2000\,QR^2 = 3617\,QR^2+1617\,Q\Delta$$

$$(7)\qquad 1728\,\Phi_{2,11} = 6PQ^3-5P^2QR+4PR^2-5Q^2R$$
$$= 6P\Delta-5R(P^2Q-2PR+Q^2),$$

where we have made use of (2) to eliminate Q^3.

Ramanujan noted that the operator

$$\vartheta = x\,d/dx$$

has the following properties when applied to $\Phi_{r,s}$, P, Q and R:—

$$\vartheta\Phi_{r,s} = \Phi_{r+1,s+1}$$
$$12\,\vartheta P = P^2-Q$$
$$3\,\vartheta Q = PQ-R$$
$$2\,\vartheta R = PR-Q^2.$$

Also

$$\vartheta\Delta = P\Delta = 12^3\sum_{n=1}^{\infty} n\tau(n)x^n$$

$$12\,\vartheta^2\Delta = (13P^2-Q)\Delta = 12^4\sum_{n=1}^{\infty} n^2\tau(n)x^n$$

$$(8)\qquad 72\,\vartheta^3\Delta = (91P^3-21PQ+2R)\Delta = 6.12^4\sum_{n=1}^{\infty} n^3\tau(n)x^n.$$

If we operate by ϑ on (6) and by ϑ^2 on (7) we obtain two further identities in which Q^3 has been replaced by $\Delta+R^2$ in accordance with (2), namely:

$$(9)\qquad 12240\,\Phi_{1,16} = (1617\,PQ-3117R)\Delta+3617\,R^2(PQ-R)$$

$$(10)\qquad 20736\,\Phi_{4,13} = (156P^3+104\,PQ-53R)\Delta-65R(P^4Q-4P^3R+6P^2Q^2-4PQR+R^2).$$

Multiplying (10) by 7 and substituting from (5) gives

$$(11)\qquad 145152\,\Phi_{4,13} = 7\Delta(156P^3+104PQ)-176R\Delta-2695680R\Phi_{4,7}.$$

We now begin to consider the above identities with respect to the moduli 7 and 49. As usual, we denote by J a power series in x with integer coefficients and write (1), (2), (5), (8), (9) and (11) as follows:

$$(1')\qquad R-1 = 7\,J$$

$$(2')\qquad 13\sum_{n=1}^{\infty}\tau(n)x^n = \Delta+49\,J$$

$$(5')\qquad \Phi_{4,7}+\Delta = 7\,J$$

$$(9')\qquad 39\,\Phi_{1,16} = 19R\Delta+40R^2(PQ-R)+49\,J$$

(11′) $$14\Phi_{4,13} = (14P^3-7PQ)\Delta+20R\Delta+6R\Phi_{4,7}+49\,J$$

and finally

(8′) $$24\,\vartheta^3\Delta = (14P^3-7PQ)\Delta+17R\Delta+49J = 18\sum_{n=1}^{\infty} n^3\tau(n)x^n+49\,J.$$

From (1′) it follows that

(1″) $$R^2 = 2R-1+49\,J$$

Multiplying both members of (9′) by 18 we have

$$\begin{aligned} R\Delta &= 34R^2(PQ-R)+33\Phi_{1,16}+49\,J \\ &= 34(2R-1)(PQ-R)+33\Phi_{1,16}+49\,J \\ &= -30R(PQ-R)+15.720\Phi_{1,4}+33\Phi_{1,16}+49\,J \\ &= 6.1584\Phi_{1,10}-18\Delta+20\Phi_{1,4}+33\Phi_{1,16}+49\,J \end{aligned}$$

by (1″), (3) and (4). Hence

(12) $$R\Delta = 31\Delta+33\Phi_{1,16}-2\Phi_{1,10}+20\Phi_{1,4}+49\,J.$$

Subtracting (11′) from (8′) gives

(13) $$18\sum_{n=1}^{\infty} n^3\tau(n)x^n = 14\Phi_{4,13}-3R\Delta-6R\Phi_{4,7}+49J.$$

Multiplying (1′) and (5′) together we get

(14) $$R\Phi_{4,7} = -R\Delta+\Delta+\Phi_{4,7}+49J.$$

Substituting (14) into (13) we find

$$18\sum_{n=1}^{\infty} n^3\tau(n)x^n = 14\Phi_{4,13}+3R\Delta-6\Delta-6\Phi_{4,7}+49J$$

(15) $$= 38\Delta+14\Phi_{4,13}+\Phi_{1,16}-6\Phi_{1,10}-6\Phi_{4,7}+11\Phi_{1,4}+49J$$

by (12). Multiplying (15) by -2 we get

(16) $$13\sum_{n=1}^{\infty} n^3\tau(n)x^n = 22\Delta+21\Phi_{4,13}-2\Phi_{1,16}+12\Phi_{1,10}+12\Phi_{4,7}+27\Phi_{1,4}+49J.$$

From (5′) we have

(17) $$21\Delta = -21\Phi_{4,7}+49J$$

Also

(18) $$\Phi_{4,13} = \sum_{n,m} n^4m^{13}x^{mn} = \sum_{m,n} n^4mx^{mn}+7J = \Phi_{1,4}+7J.$$

In view of (17), (18) and (2′), (16) can be written

(19) $$13\sum_{n=1}^{\infty} (n^3-1)\tau(n)x^n = 40\Phi_{4,7}-\Phi_{1,4}-2\Phi_{1,16}+12\Phi_{1,10}+49J.$$

Multiplying both sides of (19) by 34 and identifying coefficients of x^n on both sides of the equation we get

$$(20)\qquad (n^3-1)\tau(n) \equiv 30n\sigma_{15}(n)+16n\sigma_9(n)-(12n^4-15n)\sigma_3(n) \pmod{49}.$$

For $n = p$, a prime, (20) becomes

$$(p^3-1)\tau(p) \equiv 30p(p^{15}+1)+16p(p^9+1)-(12p^4-15p)(p^3+1) \pmod{49}.$$
$$\equiv 3p(p^3+1)(p^3-1)(10p^9-p^3-4) \pmod{49}.$$

Now if p is a non-residue of 7, as provided in the hypothesis, then $p^3 \equiv -1 \pmod 7$ so that the factor (p^3-1) is prime to 49. Moreover p^3+1 is a multiple of 7 so that the last factor $10p^9-p^3-4$ may be replaced by what it is congruent to modulo 7, namely unity. Dividing by p^3-1 therefore gives

$$\tau(p) \equiv 3p(p^3+1) \pmod{49}$$

which is the theorem.

Two immediate consequences of the theorem may be noted.

COROLLARY 1. *If p is a prime, $\tau(p)$ is divisible by 49 if and only if $p \equiv 19$, 31, or 48* (mod 49).

Proof: From (5′) follows the well-known fact that

$$\tau(n) \equiv n\sigma_3(n) \pmod 7.$$

If $n = p$, we see that $\tau(p)$ is divisible by 7 if and only if p is a non-residue of 7. Applying our theorem, for $\tau(p)$ to be divisible by 49 it is necessary and sufficient that

$$p^3 \equiv -1 \pmod{49}.$$

The three roots of this congruence are those specified in the conclusion of the corollary.

The second result is a curious congruence property for the sum

$$\Sigma_n = \sum_{\substack{u+7v=n\\ u,\, v,\, > 0}} \sigma(u)\sigma_3(7v).$$

This function occurs in an expression for $\tau(n)$ modulo 49 given by Bambah and Chowla:

$$\tau(n) \equiv 8n^4\sigma_3(n)-14\{2(1-n-n^3)\sigma_3(n)+(2n^2-3)\sigma(n)+\Sigma_n\} \pmod{49}.$$

If we take the case of n a prime p which is a non-residue of 7 and apply our theorem the following congruence results.

$$7\,\Sigma_p \equiv 5p(p+1)(p-3)(p-5) \pmod{49}.$$

In conclusion it should be noted Gupta's conjecture that if p is a prime non-residue of 7 then

$$\tau(p)/7 \equiv r-1-[3/r]-2q \pmod 7 \quad (p = 7q+r)$$

follows from our theorem by a simple consideration of each of the three cases of $r = 3$, 5 and 6.