RADIAL OSCILLATIONS OF A GASEOUS STAR OF POLYTROPIC INDEX I.

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SUMMARY.

The periods of the fundamental, first, and second modes of pulsations have been found out for a star of polytropic index $n = 1$.

The radial oscillations of a gaseous star of the polytrope $n = 3$ (standard model) were first investigated by Eddington (1926). His method consisted in numerical integration of the pulsation equation by finite differences, to find the period of the adiabatic radial oscillation of the standard model. Recently Schwarzschild (1941) with much more greater accuracy has found out the periods of the first four spherically symmetrical overtone pulsations for the standard model. Miller (1929) extended the calculations to the models of polytrope $n = 2$, and $n = 4$, his work was based on the single value $\alpha = 0{\cdot}2$, which corresponds to the ratio of specific heats, $\gamma = 10/7$. In the present note a similar investigation is made for the polytrope $n = 1$ to find the periods of the fundamental, first, and second modes for $\alpha = 0{\cdot}6$, which corresponds to $\gamma = 5/3$, and also the periods of the fundamental mode for $\alpha = 0{\cdot}4$ and $\alpha = 0{\cdot}2$, which corresponds to $\gamma = \frac{20}{13}$ and $\gamma = \frac{10}{7}$ respectively.

The differential equation for small adiabatic radial oscillation, as given by Eddington, is

$$\frac{d^2f}{dz^2}+\left(\frac{4-\mu}{z}\right)\frac{df}{dz}+\left(\frac{\omega^2}{u}-\frac{\alpha\mu}{z^2}\right)f = 0 \quad \cdots \quad \cdots \quad \cdots \quad (1)$$

where f is the amplitude of the displacement $\frac{d\xi}{\xi}$, z is proportional to ξ, mean distance from the centre, $\alpha = 3-\frac{4}{\gamma}$, γ being the ratio of specific heats, u is the Emden function of polytropic index $n = 1$, $\mu = \frac{g_0\rho_0 z}{P_0}$, g_0, ρ_0, P_0, being respectively the mean values of gravity, density, and pressure at a distance z from the centre, $\omega^2 = \nu^2(2\pi G\rho_c\gamma)^{-1}$, where the period $\Pi = \frac{2\pi}{\nu}$, G, the constant of Gravitation and ρ_c the mean density at the centre. Equation (1) has a solution non-singular both at $z = 0$, and z at the surface of the star.

For the polytrope $n = 1$, the well-known Emden equation is

$$\frac{d^2u}{dz^2}+\frac{2}{z}\frac{du}{dz}+u = 0$$

which gives $u = \frac{\sin z}{z}$ as a solution. From Emden's theory we have

$$\mu = -(n+1)\frac{z}{u}\frac{du}{dz}, \text{ for } n = 1, \ \mu = -2\frac{z}{u}\frac{du}{dz}, \text{ i.e. } \mu = -2(z \cot z-1).$$

With this value of μ the pulsation equation (1) takes the form

$$\frac{d^2f}{dz^2}+\frac{2(1+z\cot z)}{z}\frac{df}{dz}+\left\{\frac{\omega^2 z}{\sin z}+\frac{2\alpha(z\cot z-1)}{z^2}\right\}f=0. \quad .. \quad .. \quad (2)$$

This equation has been solved numerically. All the calculations have been done with the help of Mathematical Tables I (1946) of the British Association for the Advancement of Science, and the calculating machine. Table I gives the resulting characteristic values of ω^2. Table II gives the amplitudes f of the displacements

TABLE I.

Characteristic values of ω^2.

Mode.	$\alpha = 0{\cdot}6$.	$\alpha = 0{\cdot}4$.	$\alpha = 0{\cdot}2$.
0	0·231	0·155	0·078
1	1·517		
2	3·580		

TABLE II.

Amplitudes of the Displacements, f.

z	Fundamental. $\alpha = 0{\cdot}6$.	First Overtone. $\alpha = 0{\cdot}6$.	Second Overtone. $\alpha = 0{\cdot}6$.
0·0	+1·000000	+1·000000	+1·000000
0·1	1·000169	0·998882	0·996820
0·2	1·000677	0·995518	0·987281
0·3	1·001525	0·989874	0·971386
0·4	1·002716	0·981890	0·949136
0·5	1·004251	0·971495	0·920555
0·6	1·006143	0·958596	0·885671
0·7	1·008389	0·943052	0·844515
0·8	1·011009	0·924713	0·797168
0·9	1·014006	0·903389	0·743728
1·0	1·017390	0·878859	0·684350
1·1	1·021175	0·850865	0·619257
1·2	1·025374	0·819105	0·548757
1·3	1·030004	0·783221	0·473277
1·4	1·035102	0·742868	0·393396
1·5	1·040853	0·697654	0·309941
1·6	1·047084	0·646920	0·223844
1·7	1·053810	0·590042	0·136419
1·8	1·061066	0·526318	+0·049379
1·9	1·068886	0·454939	−0·035065
2·0	1·077309	0·374974	−0·114079
2·1	1·086374	0·285357	−0·184050
2·2	1·096138	0·184876	−0·240444
2·3	1·106766	+0·072271	−0·277903
2·4	1·118204	−0·054152	−0·288854
2·5	1·130496	−0·196154	−0·264401
2·6	1·143708	−0·355598	−0·193931
2·7	1·157893	−0·534282	−0·065351
2·8	1·173099	−0·733397	+0·133422
2·9	1·189288	−0·951614	+0·408698
3·0	1·206063	−1·175893	+0·734701
3·1	+1·218532	−1·251806	+0·780444

as functions of mean distance from the centre for the fundamental, first and second mode (for $\alpha = 0{\cdot}6$). Figure I shows the amplitudes f of the displacements as functions of mean distance z from the centre for the fundamental, first, and second mode.

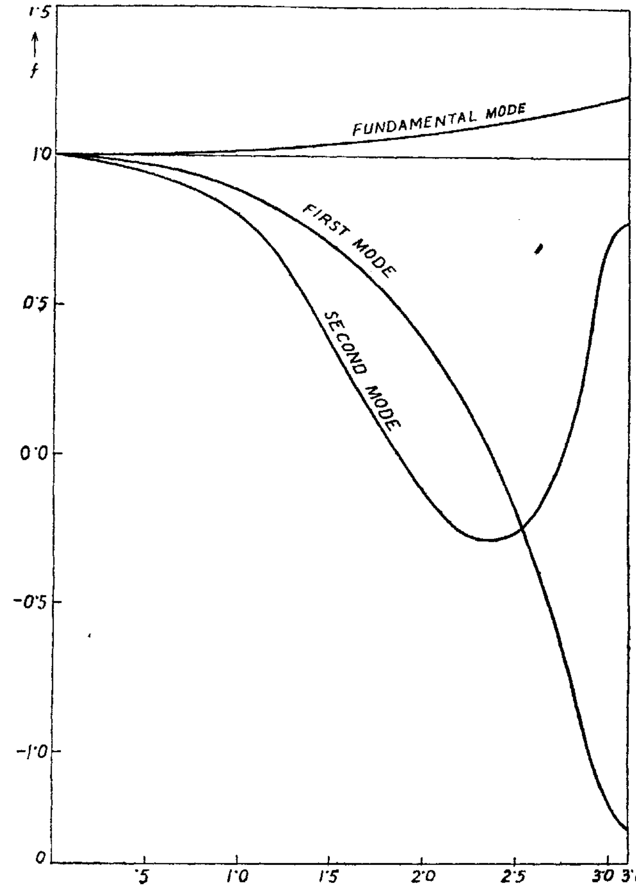

FIG. 1.—The amplitudes f of the displacements in the fundamental and the first two modes for $\alpha = 0{\cdot}6$, as functions of the mean distance z from the centre.

The characteristic values of ω^2 were determined in the following way. A trial value of ω^2 was chosen for $\alpha = 0{\cdot}6$. For the neighbourhood of $z = 0$ and z at the surface of the star, which in our case is $z = \pi$, equation (2) was solved by power series; the series developed at $z = 0$ (neglecting eighth and higher powers) was used to compute f at $z = 0$, $0{\cdot}1$, $0{\cdot}2$, $0{\cdot}3$ respectively, and with these starting values the integrations of the equation (2) have been performed numerically starting with the arbitrary amplitude $f = 1$ at the centre and the solution is continued towards the surface until it becomes evident how it is going to behave at the surface itself. The condition for the Node to fall at the surface of the star being

$$3f + z\frac{df}{dz} = 0.$$

As the adiabatic approximation breaks down near the boundary we have not strictly followed this condition. After a few successive trials the characteristic value of

ω^2 for the fundamental mode for $\alpha = 0{\cdot}6$ was located between 0·22 and 0·24. This approximation for ω^2 was utilised to obtain the characteristic value of ω^2 correct up to third place in the subsequent integrations and fair accuracy has been obtained with the help of interpolation. We have followed Adam's method of integration as sketched by Von Zeipel (1924) in one of his papers in doing all the integrations. For the correct value of ω^2 it has been observed that the displacement functions show a steady increase in going from the centre right up to the surface of the star for the fundamental mode. Adjacent solutions have been found to show signs of running off to positive or negative infinity. We have found that a lower value of ω^2 would give abrupt increase in the displacement function at the surface of the star, whereas a higher value of ω^2 would give abrupt decrease in the displacement function at the surface of the star. This idea has greatly facilitated in arriving at the correct characteristic values of ω^2 for the fundamental mode. Similar processes with some modification have been followed for finding out the characteristic values of ω^2 for the first and second modes for $\alpha = 0{\cdot}6$.

We have found the periods of pulsations for the first and second modes as we shall study the Anharmonic pulsations for the polytrope $n = 1$ in the next paper, which is under preparation.

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