ON THE ZEROS OF A CERTAIN POLYNOMIAL.

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1. Recently Venkataraman (1950) has generalized a result of Prof. K. S. K. Iyer and has shown that the zeros of any section of the power series for e^z, say

$$\frac{1}{r\,!}+\frac{z}{(r+1)\,!}+\ldots\ldots+\frac{z^n}{(r+n)\,!}$$

lie in the region $(r+1) \leqslant |z| \leqslant (r+n)$.

In an attempt to generalize this result, I have proved a similar property for any section of the power-series for $\left(\frac{e^x-1}{x}\right)^r$ and the power-series

$$\sum_{n=0}^{\infty} (n+\alpha)^r \frac{x^n}{n\,!}.$$

The object of this note is to prove a similar theorem when formally

$$\left(\frac{(1-x)^{-h}-1}{x}\right)^r = \sum_{n=0}^{\infty} p_n(r)x^n, \qquad (h>1) \qquad \ldots \quad \ldots \quad (1)$$

and r is a positive integer.

I shall prove that the zeros of

$$P(z) = \sum_{k=0}^{n} p_{m+k}(r)z^k$$

lie in the region

$$\frac{p_m(r)}{p_{m+1}(r)} \leqslant |z| \leqslant \frac{p_{m+n-1}(r)}{p_{m+n}(r)}$$

2. In order to prove this result, we shall first prove the following lemmas :—

Lemma 1. The functions $p_n(r)$ satisfy the following relation—

$$(n+r)p_n(r) = (n+r+hr-1)p_{n-1}(r)+hrp_n(r-1), \qquad (n \geqslant 1) \qquad \ldots \quad (2)$$

This is proved easily by multiplying (1) by x^r and differentiating. A little rearrangement gives (2).

Lemma 2. The functions $p_n(r)$ satisfy the inequality

$$\frac{p_n(r)}{p_{n+1}(r)} < \frac{p_{n+1}(r)}{p_{n+2}(r)}, \qquad \text{for } \begin{array}{l} n = 1, 2, \ldots\ldots; \\ r = 1, 2, \ldots\ldots \end{array} \qquad \ldots \quad \ldots \quad (3)$$

To prove this we observe that from the known relations

$$p_0(r) = h^r, \quad p_n(1) = \binom{n+h}{n+1}$$

$$p_n(r+1) = \sum_{\lambda=0}^{n} p_\lambda(r) p_{n-\lambda}(1)$$

we can prove the inequality (3) by induction on r. For $r = 1$, the inequality is easily verified.

3. *Proof of the main Theorem :*

(i) Write

$$\alpha = p_m(r)/p_{m+1}(r).$$

Then

$$(\alpha - z)P(z) = \alpha p_m(r) - \left[\sum_{k=0}^{n-1} \{ p_{m+k}(r) - \alpha p_{m+k+1}(r) \} z^{k+1} + p_{m+n}(r) z^{n+1} \right].$$

By Lemma 2,

$$p_{m+k}(r) - \alpha p_{m+k+1}(r) > 0,$$

so that for $|z| < \alpha$,

$$\left| \sum_{k=0}^{n-1} \{ p_{m+k}(r) - \alpha p_{m+k+1}(r) \} z^{k+1} + p_{m+n}(r) z^{n+1} \right| < \alpha p_m(r).$$

Hence

$$|(\alpha - z)P(z)| > 0$$

i.e. all the zeros of $P(z)$ lie in $|z| \geqslant \alpha$.

(ii) Now put

$$\beta = p_{m+n-1}(r)/p_{m+n}(r)$$

and let

$$g(z) = z^n P\left(\frac{\beta}{z}\right) = b_0 z^n + b_1 z^{n-1} + \ldots\ldots + b_n.$$

Then for $|z| < 1$,

$$|(1-z)g(z)| > b_n - |\{(b_n - b_{n-1})z + \ldots\ldots + (b_1 - b_0)z^n + b_0 z^{n+1}\}| > 0,$$

since by lemma 2, $b_r > b_{r-1}$, $(r = 1, 2, \ldots\ldots, n-1)$ and $b_n = b_{n-1}$. Thus $|g(z)| > 0$ if $|z| < 1$ and hence $|P(z)| > 0$ if $|z| > \beta$.

Hence the zeros of $P(z)$ lie in $\alpha \leqslant |z| \leqslant \beta$.

4. Lastly in a similar order of ideas, we shall state the following theorem :—

Assume that a_m, c_m are each > 0, $n \geqslant 0$, and that

$$a_m^2 > a_{m-1} a_{m+1}, \quad c_m^2 > c_{m-1} c_{m+1}, \qquad \text{for } m \geqslant 1.$$

Put

$$\left(\sum_{m=0}^{\infty} a_m z^m\right)\left(\sum_{m=0}^{\infty} c_m z^m\right) = \sum_{m=0}^{\infty} A_m z^m.$$

Then for $m = 1, 2, \ldots.$

$$A_m^2 > A_{m-1} A_{m+1}.$$

This theorem is easily proved on observing that

$$A_m = \sum_{\lambda=0}^{m} a_{m-\lambda} c_\lambda.$$

This theorem is more general than a Lemma of Polya (1950).

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Abstract.

An attempt is made in this paper to consider the zeros of sections of the power series expansion of $\left(\frac{(1-x)^{-h}-1}{x}\right)^r$ where r is a positive integer and $h > 1$. The result is connected with an interesting theorem of Polya.

References.

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