ON LATTICE COVERINGS

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1. Introduction.

Let $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ be two points in the n-dimensional Euclidean space R_n. We shall use the usual vector notation to denote the points $(x_1+y_1, \ldots, x_n+y_n)$ and $(\lambda x_1, \ldots, \lambda x_n)$, λ real, by the symbols $X-Y$ and λX, respectively. If S and T are two sets of points in R_n, we shall use the symbol $\lambda S+\mu T$ for the set of points $\lambda X+\mu Y$ with the points X and Y lying in S and T respectively. By $|X|$ we shall mean $(x_1^2+ \ldots +x_n^2)^{\frac{1}{2}}$, the distance of X from the origin O.

Let K be any bounded or unbounded set and let Λ be a lattice of points $X = (x_1, \ldots, x_n)$, where

$$x_h = \sum_{k=1}^{n} a_{hk}u_k; \quad h = 1, \ldots, n; \; u_k = 0, \pm 1, \pm 2, \ldots$$

and the coefficients a_{hk} are real and the determinant $d(\Lambda) = \left|\left|a_{hk}\right|\right| \neq 0$. We shall call Λ a *covering lattice* for K if every point of R_n belongs to the set $K-\Lambda$.

The upper bound of the determinants $d(\Lambda)$ extended over all covering lattices Λ for K will be called the *covering constant* of K and will be denoted throughout by $c(K)$ or by c when there is no danger of confusion.

Let K be a closed bounded set with non-zero Jordan measure $V(K)$ and let B be a closed bounded convex body with Jordan volume $V(B)$. We say that there is a *lattice covering of B by u sets K* if there exists a lattice Λ (not necessarily homogeneous)[2] such that every point of B belongs to the set $K+\Lambda$ and such that there are exactly u sets $K+X$, with X a point of Λ, which have a point in common with B.

Let $u(B, K)$ denote the smallest number u for which there is a lattice covering of B by u sets K. Then the *density* $\vartheta(B, K)$ *of the most economical lattice covering of B by K* is defined by the relation

$$\vartheta(B, K) = u(B, K)\; V(K)/V(B) \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

When K is a convex body a result of Hlawka (1949, Satz 1 and Satz 26 folgerung) states that $\lim\limits_{\lambda \to \infty} \vartheta(\lambda B, K)$ exists and is in fact equal to $V(K)/c(K)$, independent of the choice of B. By a slight modification of Hlawka's argument it is easy to prove that this result is valid for any closed bounded set K with an inner point. This limit will be called the *density of the best lattice covering of space by K* and will be denoted by the symbol $\vartheta(K)$.

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[2] i.e. Λ may be a set of the type Λ^*+A, where Λ^* is a lattice of the type defined earlier and A is a point not in Λ^*.

A few years ago, Hlawka (1949, Satz 2) proved that for any closed bounded convex body K with a centre, we have

$$\vartheta(K) \leqslant n^n. \qquad (2)$$

Rogers (1950) improved the above to

$$\vartheta(K) \leqslant 3^{n-1}. \qquad (3)$$

Bambah and Roth (1952) have shown that if K is further supposed to be symmetrical about the co-ordinate plane, then

$$\vartheta(K) \leqslant \frac{\pi}{3\sqrt{3}} \frac{n^n}{n!} \sim e^n \left(\frac{\pi}{54n}\right)^{\frac{1}{2}} \text{ as } n \to \infty. \qquad (4)$$

When K is a sphere Davenport (1952) has proved that

$$\vartheta(K) < (1{\cdot}15)^n, \qquad (5)$$

while Bambah and Davenport (1952) have proved that

$$\vartheta(K) > \tfrac{4}{3} - \epsilon_n, \qquad (6)$$

where $\epsilon_n \to 0$ as $n \to \infty$.

If K is a bounded convex body in two dimensions, it follows from certain results of Fary (1950) that $c(K)$ is equal to the area of the largest symmetrical hexagon contained in K.

Our object in this paper is to prove some general results about $c(K)$ and certain related lattices for sets K, belonging to more general types than that of closed, bounded and convex bodies.

It may be relevant to remark that there is a close relation between covering lattices and certain problems in the Theory of Numbers. For example, Davenport (1951) has recently obtained the following remarkable theorem.

Let $L_1(x, y) = \alpha x + \beta y$ and $L_2(x, y) = \gamma x + \delta y$ be two linear forms with real coefficients and determinant $\Delta = \left|\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}\right| \neq 0$. Then there exist real numbers x_0, y_0 such that

$$|L_1(x+x_0, y+y_0)\, L_2(x+x_0, y+y_0)| > \frac{1}{128}\Delta \qquad (7)$$

for all integers (x, y).

In the covering notation the above theorem can be rewritten as follows. Let K be the body $|xy| \leqslant 1$. Then $c(K) < 128$. Prasad (unpublished) and Cassels (1952) have proved (7) with better constants.

2. K CLOSED AND BOUNDED.

In this section our main object is to prove—

THEOREM 1: *Let K be a closed bounded set with the origin O in its interior. Then there exists a covering lattice Λ for K such that*

$$d(\Lambda) = c(K). \qquad (8)$$

As O is an inner point of K, it contains a sphere C_t with the centre at the origin. Since every covering lattice of C_t is also a covering lattice for K, we have

$$c = c(K) \geqslant c(C_t) > 0. \qquad (9)$$

On the other hand, K being bounded is contained in a finite sphere C_e with the centre at the origin so that

$$c(K) \leqslant c(C_e) \leqslant V(C_e), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

where $V(C_e)$ is the volume of C_e. Thus c is a non-zero finite number.

By the definition of c, there exists an infinite sequence $\Lambda^{(1)}, \Lambda^{(2)}, \ldots$ of covering lattices (not necessarily all different) of K such that $d(\Lambda^{(i)}) \to c$ as $i \to \infty$. Without loss of generality we can suppose that for each i

$$\frac{c}{2} < d(\Lambda^{(i)}) \leqslant c. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

It was shown by Mahler (1946, theorem 1) that every lattice Λ in R_n has a reduced basis $Y_1, \ldots, Y_n$ such that

$$| Y_1 | \ldots | Y_n | \leqslant \gamma_n d(\Lambda), \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

where γ_n is a constant depending only on n. Let $Y_1^{(i)}, \ldots, Y_n^{(i)}$ be such a reduced basis of the lattice $\Lambda^{(i)}$ defined above. Then using the inequalities (11) and (12) we have

$$\frac{c}{2} < d(\Lambda^{(i)}) \leqslant | Y_1^{(i)} | \ldots | Y_n^{(i)} | \leqslant \gamma_n d(\Lambda^{(i)}) \leqslant c\gamma_n. \quad \ldots \quad \ldots \quad (13)$$

We now prove

LEMMA 1: *For each i and each $k = 1, \ldots, n$ we have*

$$| Y_k^{(i)} | < 4\gamma_n \rho, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

where ρ is the radius of the smallest sphere with the centre at the origin that contains the set K.

Proof: Suppose, for example,

$$| Y_n^{(i)} | \geqslant 4\gamma_n \rho.$$

Then using relation (13) we have

$$| Y_1^{(i)} | \ldots | Y_{n-1}^{(i)} | \leqslant \frac{c}{4\rho}. \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

Now let $P^{(i)}$ be the fundamental cell of $\Lambda^{(i)}$ spanned by $Y_1^{(i)}, \ldots, Y_n^{(i)}$. Let Π be the face containing $O, Y_1^{(i)}, \ldots, Y_{n-1}^{(i)}$. Let λ be the perpendicular distance of Y_n from Π. Then, using (13) and (15), we get

$$\frac{c}{2} < d(\Lambda^{(i)}) = V(P^{(i)}) \leqslant \lambda | Y_1^{(i)} | \ldots | Y_{n-1}^{(i)} | \leqslant \lambda \frac{c}{4\rho}; \quad \ldots \quad \ldots \quad (16)$$

here $V(P^{(i)})$ denotes the volume of $P^{(i)}$. From (16) it is clear that

$$\lambda > 2\rho. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

Let Π_1 and Π_2 be the hyperplanes parallel to Π at distances ρ and $\lambda - \rho$ from it. Then P' the part of $P^{(i)}$ intercepted between Π_1 and Π_2 has a non-zero volume $V(P')$. Since $\Lambda^{(i)}$ is a covering lattice for K and so also for the sphere $C: \frac{|X|}{\rho} \leqslant 1$, which contains K, every point of P' lies in the set of spheres $C+\Lambda^{(i)}$. Therefore

$$\begin{aligned} 0 < V(P') &\leqslant \sum_{A \in \Lambda^{(i)}} V\{(C+A) \cap P'\} \\ &= \sum_{A \in \Lambda^{(i)}} V\{C \cap (P'+A)\} \\ &\leqslant V_1 + V_2, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18) \end{aligned}$$

where V_1 is the volume of the part of C which lies to the side of Π_1 away from O and V_2 is the volume of the part of C to the side of $-\Pi_1$ away from O. As the distance of Π_1 from O is equal to ρ, while C is a sphere of radius ρ, it is clear that both V_1 and V_2 are equal to zero, which is in contradiction to (18). This proves the lemma.

By (11) and (14) it follows from a result of Mahler (1946, Theorem 2) that there exists an infinite subsequence of $\{\Lambda^{(i)}\}$ which converges to a lattice Λ with $d(\Lambda) = c(K) = c$. Without loss of generality we can suppose that the sequence $\{\Lambda^{(i)}\}$ itself converges to the lattice Λ. We now complete the proof of Theorem 1 by proving that Λ is a covering lattice for K.

Let Z be any point in R_n and let L be a lattice. We observe that Z lies in the set $K+L$ if and only if L has a point in the set $Z+\bar{K}$, where the set $\bar{K}$ is the image of K in the origin O.

Now, since each $\Lambda^{(i)}$ is a covering lattice for K, it follows that for any point Z in R_n each $\Lambda^{(i)}$ has a point $X^{(i)}$, say, in the set $Z+\bar{K}$. As $Z+\bar{K}$ is closed and bounded, it is clear that there is an infinite subsequence of $\{X^{(i)}\}$ which converges to a point X lying in the set $Z+\bar{K}$. Because of the boundedness of the lattices $\Lambda^{(i)}$ it is easy to see that X must be a point of Λ. Thus for every point Z in R_n, Λ has a point X in the set $Z+\bar{K}$ and the theorem follows.

DEFINITION: If Λ is a covering lattice for a set K and if $d(\Lambda) = c(K)$, then we shall call Λ a *maximal covering lattice* for K.

We next prove

THEOREM 2: *If K is a closed and bounded set and if Λ is a maximal covering lattice for K, then K has n independent points on its boundary which are just covered, i.e., which do not lie in the interior of any $K+X$ with X in Λ.*

Proof: Suppose K does not contain n independent points which are just covered. Denote by Π the set of all points of K which are just covered. Let L be the linear space of lowest dimension $f(0 \leqslant f \leqslant n-1)$ which contains Π. Let S be the set of all points X_i of Λ for which the set $K+X_i$ contains a point of Π. Then the set S must lie in L, for otherwise we have a contradiction to the assumption that all points of K which are just covered lie in L. Let M be the linear space of lowest dimension $g(0 \leqslant g \leqslant f \leqslant n-1)$ which contains S. Then by Minkowski's method of adoption of lattices we can find a basis $Y_1, \ldots, Y_n$ of Λ such that $Y_1, \ldots, Y_g$ lie in M and generate it, while $Y_{g+1}, \ldots, Y_n$ lie outside M. Let P be the fundamental cell, together with its boundary, spanned by $Y_1, \ldots, Y_n$. Then any point A of P is either completely covered (i.e. is an inner point of a set $K+X$, X in Λ) or is congruent mod. Λ to a point B of Π.

We now observe that since K is bounded, any point B of Π is at a distance greater than a positive number $r(B)$ from all sets $K+X$, where X is a point of Λ not in the set S. Consequently, since Λ is a covering lattice for K, every point in the open sphere of radius $r(B)$ about B must lie in a set of the type $K+X$, where X is a point of Λ lying in S.

Let Σ be the set of all points $\Xi = (\xi_1, \ldots, \xi_n)$, where $0 \leqslant \xi_i \leqslant 1$, for all $i = 1, \ldots, n$. Since the point $\xi_1 Y_1 + \ldots + \xi_n Y_n$ is a point of P, it is either an inner point of a set $K+X$, with X a point of Λ, or it is congruent mod. Λ to a point B of Π. We can consequently associate with every point Ξ of Σ an open sphere $c(\Xi)$ containing Ξ and a real number $\epsilon(\Xi) > 0$ such that for all positive $\epsilon < \epsilon(\Xi)$ the points $\xi_1 Y_1 + \ldots + \xi_{n-1} Y_{n-1} + \xi_n Y'_n$ for all $(\xi_1, \ldots, \xi_n)$ in $c(\Xi)$ lie in the set $K+\Lambda'$ where $Y'_n = Y_n(1+\epsilon)$ and Λ' is the lattice generated by $Y_1, \ldots, Y_{n-1}, Y'_n$. From this it follows easily by the Heine-Borel Theorem that there exists a positive number ϵ^* such that the lattice Λ^* generated by $Y_1, \ldots, Y_{n-1}, Y_n(1+\epsilon^*)$ is a covering lattice for K. But $d(\Lambda^*) = (1+\epsilon^*)d(\Lambda) = (1+\epsilon^*)c(K) > c(K)$. Thus we get a contradiction which proves the theorem.

3. K CLOSED BUT UNBOUNDED.

In this section we suppose K is an unbounded closed set which contains the origin in its interior and which has Lebesgue measure $V(K)$. We first prove the almost trivial

THEOREM 3: $c(K) \leqslant V(K)$.

Proof: If $V(K) = \infty$, there is nothing to prove. Therefore, we can suppose $V(K) < \infty$. Let Λ be a covering lattice for K and P a fundamental cell of Λ. Since every point of P lies in the set $K+\Lambda$, we clearly have

$$\begin{aligned} d(\Lambda) = V(P) &\leqslant \sum_{A \epsilon \Lambda} V\{(K+A) \cap P\} \\ &= \sum_{A \epsilon \Lambda} V\{K \cap (P+A)\} \\ &= V(K) \cap R_n) \\ &= V(K), \end{aligned}$$

and the theorem follows.

In the rest of this section we shall suppose $V(K) < \infty$, and our object will be to prove

THEOREM 4: *Let K be an unbounded but closed set containing O in its interior and having Lebesgue measure $V(K) < \infty$. Then there exists a lattice Λ with $d(\Lambda) = c(K)$ such that the set of all points of R_n which do not lie in $K+\Lambda$ has Lebesgue measure zero.*

We shall in the rest of the paper write $K^{(t)}$ for the set of all points of K which lie in the sphere $|X| \leqslant t$. We shall use the symbol R_t both for $V(K) - V(K^{(t)})$ and for the set $K \cap CK^{(t)}$, i.e. for the set of points of K which do not lie in $K^{(t)}$. In the usual notation we shall write $\delta(X, S)$ for the distance, i.e. $\underset{Y \epsilon S}{bd} |X - Y|$, of the point X from the set S.

If S is closed, it is clear that $X \epsilon S$ if and only if $\delta(X, S) = 0$. Further, Λ is a covering lattice for S if and only if for every point A of R_n there is a point X of Λ, such that

$$\delta(A, K+X) = \delta(A-X, K) = 0.$$

Since O is an inner point of K, there is a sphere C: $|X| \leqslant \rho$, which is contained in K. Consequently

$$0 < c(C) \leqslant c(K) \leqslant V(K) < \infty.$$

Thus $c = c(K)$ is a non-zero finite number.

As in §2 there exists an infinite sequence $\Lambda^{(1)}, \Lambda^{(2)}, \ldots$ of covering lattices of K such that $d(\Lambda^{(i)}) \to c$ and such that

$$\frac{c}{2} < d(\Lambda^{(i)}) \leqslant c. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

Also $\Lambda^{(i)}$ has a reduced basis $Y_1^{(i)}, \ldots, Y_n^{(i)}$, such that

$$\frac{c}{2} < d(\Lambda^{(i)}) \leqslant |Y_1^{(i)}| \ldots |Y_n^{(i)}| \leqslant \gamma_n d(\Lambda^{(i)}) \leqslant c\gamma_n, \ldots \quad \ldots \quad (20)$$

where γ_n is a constant depending only on n.

It is well known that $R_t = V(K) - V(K^{(t)}) \to 0$ as $t \to \infty$. Let $t = t_1$ be a fixed number for which $R_{t_1} < \frac{1}{6} c$. We now prove a lemma (lemma 2) very similar to lemma 1 of §2.

Lemma 2: *For each i and each $k = 1, \ldots, n$, we have*

$$|Y_k^{(i)}| < 6\gamma_n t_1. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

Proof: Suppose, for example, $|Y_n^{(i)}| \geqslant 6\gamma_n t_1$.
Then from (20), we have

$$|Y_1^{(i)}| \ldots |Y_{n-1}^{(i)}| \leqslant \frac{1}{6}\frac{c}{t_1}. \qquad \ldots \quad \ldots \quad \ldots \quad (22)$$

Now let $P^{(i)}$ be the fundamental cell of $\Lambda^{(i)}$ spanned by $Y_1^{(i)}, \ldots, Y_n^{(i)}$. Let Π be its face containing the points O, $Y_1^{(i)}, \ldots, Y_{n-1}^{(i)}$ and let $V(\Pi)$ be the $(n-1)$-dimensional volume of Π. Let λ be the perpendicular distance of Y_n from Π. Then, by (20) and (22), we have

$$\begin{aligned} \frac{c}{2} < d(\Lambda^{(i)}) = V(P^{(i)}) &= \lambda V(\Pi) \\ &\leqslant \lambda |Y_1^{(i)}| \ldots |Y_{n-1}^{(i)}| \\ &\leqslant \frac{\lambda c}{6t_1}, \end{aligned}$$

so that

$$\lambda > 3t_1. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

Let Π_1 and Π_2 be the hyper-planes through the points $\frac{1}{3} Y_n^{(i)}$, $\frac{2}{3} Y_n^{(i)}$, parallel to Π. Let P' be the part of $P^{(i)}$ intercepted between Π_1 and Π_2. Then

$$V(P') = \frac{1}{3} V(P^{(i)}) = \frac{1}{3} d(\Lambda^{(i)}). \qquad \ldots \quad \ldots \quad \ldots \quad (24)$$

Since $\Lambda^{(i)}$ is a covering lattice for K, every point of P' lies in the set $K + \Lambda^{(i)}$. Consequently, we have

$$\begin{aligned} \frac{1}{6} c < \frac{1}{3} d(\Lambda^{(i)}) &= V(P') \\ &\leqslant \sum_{A \in \Lambda^{(i)}} V\{(K+A) \cap P'\} \\ &= \sum_{A \in \Lambda^{(i)}} V\{K \cap (P'+A)\} \\ &\leqslant V_1 + V_2, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (25) \end{aligned}$$

where V_1 is the volume of the part of P' which lies on the side of Π_1 away from O and V_2 is the part of P' which lies on the side of $-\Pi_1$ away from O. Since the distance of Π_1 from O is greater than t_1, it is clear that $V_1+V_2 \leqslant R_{t_1}$, so that, by (25) and the definition of t_1, we have

$$\frac{1}{6}c < V_1+V_2 \leqslant R_{t_1} < \frac{1}{6}c,$$

which is a contradiction that establishes the lemma.

From Lemma 2, it follows as in §2 that there exists an infinite subsequence of $\{\Lambda^{(i)}\}$ converging to a lattice Λ with $d(\Lambda) = c$. Without loss of generality we can suppose that the sequence $\{\Lambda^{(i)}\}$ itself converges to Λ. Further $Y_1, \ldots, Y_n$, where

$$Y_k = \lim_{i\to\infty} Y_k^{(i)},$$

is a reduced basis of Λ.

From inequalities (20) and (21) it is easy to see that the parallelopiped $\bar{P}$ whose vertices are given by $\pm Y_1^{(i)} \pm \ldots \pm Y_n^{(i)}$ contains the sphere $|X| \leqslant \frac{c}{2}\left(\frac{1}{6\gamma_n t_1}\right)^{n-1}$ in its interior. As this sphere is independent of i, we easily deduce that given a fixed sphere C, there are only a finite number of sets of integers $(a_1, \ldots, a_n)$ such that there exists a $+$ve integer i for which the point $a_1Y_1^{(i)}+\ldots+a_nY_n^{(i)}$ lies in C. Again, since $\bar{P}$ does not contain any point of $\Lambda^{(i)}$, except O, in its interior, it is easy to see that the lattices $\Lambda^{(i)}$ are bounded in the sense of Mahler (1946, Def. 1).

Since both $K^{(t)}$ and $\Lambda^{(i)}$ are bounded it is easy to verify that for every point X of $P^{(i)}$, $\underset{A\epsilon\Lambda^{(i)}}{bd}\ \delta(X, K^{(t)}+A)$ is actually attained. Further, if C is a big enough sphere depending only on t and c, the bd is attained at a point $A^{(i)}$ of $\Lambda^{(i)}$ which lies in C. Let C be such a sphere. Let S be the set of integers $(a_1, \ldots, a_n)$ such that for some $i = 1, 2, \ldots$ the point $a_1Y_1^{(i)}+\ldots+a_nY_n^{(i)}$ lies in C. As already observed S has only a finite number of its members. Therefore for every X in $P^{(i)}$ we have

$$\underset{A\epsilon\Lambda^{(i)}}{bd}\ \delta(X, K^{(t)}+A) = \min_{(a_1,\ldots,a_n)\epsilon S} \delta(X-a_1Y_1^{(i)}-\ldots-a_nY_n^{(i)}, K^{(t)}). \tag{26}$$

Now define E_i to be the set of points $(\xi_1, \ldots, \xi_n)$, $0 \leqslant \xi_k \leqslant 1$, $k = 1, \ldots, n$, such that for every $j \geqslant 1$ the point $\xi_1Y_1^{(j)}+\ldots+\xi_nY_n^{(j)}$ does not lie in $K^{(t)}+\Lambda^{(j)}$. Similarly define E as the set of points $(\xi_1, \ldots, \xi_n)$, $0 \leqslant \xi_k \leqslant 1$, $k = 1, \ldots, n$, for which the point $\xi_1Y_1+\ldots+\xi_nY_n$ does not lie in $K^{(t)}+\Lambda$. Then we have

Lemma 3: $E \subseteq E_1 \cup E_2 \cup \ldots$

Proof: Let $(\xi_1, \ldots, \xi_n)\epsilon E$. Then $\xi_1Y_1+\ldots+\xi_nY_n$ does not lie in $K^{(t)}+\Lambda$. Therefore,

$$\delta\{(\xi_1Y_1+\ldots+\xi_nY_n)-(a_1Y_1+\ldots+a_nY_n), K^{(t)}\} > \epsilon > 0$$

for some ϵ and all $(a_1, \ldots, a_n)$ in S, i.e. all the points $(\xi_1-a_1)Y_1+\ldots+(\xi_n-a_n)Y_n$ are exterior points of $K^{(t)}$. Since there are only a finite number of members of S and since $Y_k^{(i)} \to Y_k$ it follows that there is a j, such that for all $i \geqslant j$ and all $(a_1, \ldots a_n)$ in S,

$$\delta\{(\xi_1Y_1^{(i)}+\ldots+\xi_nY_n^{(i)})-(a_1Y_1^{(i)}+\ldots+a_nY_n^{(i)}), K^{(t)}\} > 0.$$

This means that $\xi_1 Y_1^{(i)}+\ldots+\xi_n Y_n^{(i)}$ does not lie in $K^{(t)}+\Lambda^{(i)}$ for all $i \geqslant j$, i.e. $(\xi_1, \ldots, \xi_n)\epsilon E_j$. From this the lemma follows at once

Lemma 4: *Let $F^{(i)}$ be the set of points $(\xi_1, \ldots, \xi_n)$, $0 \leqslant \xi_k \leqslant 1$, for which the corresponding points $\xi_1 Y_1^{(i)}+\ldots+\xi_n Y_n^{(i)}$ do not lie in $K^{(t)}+\Lambda^{(i)}$. Then*

$$m_e(F^{(i)}) \leqslant R_t \cdot \frac{1}{d(\Lambda^{(i)})} < \frac{2}{c} R_t, \qquad \ldots \quad \ldots \quad \ldots \quad (27)$$

where $m_e(F^{(i)})$ denotes the outer Lebesgue measure of $F^{(i)}$.

Proof: Since $\Lambda^{(i)}$ is a covering lattice for K, the parallelopiped $P^{(i)}$ spanned by $Y_1^{(i)}, \ldots, Y_n^{(i)}$ is contained in $K+\Lambda^{(i)}$. Consequently the points of $P^{(i)}$ which are not contained in $K^{(t)}+\Lambda^{(i)}$ are contained in $R_t+\Lambda^{(i)}$. Therefore, if we denote by $G^{(i)}$ the set of points of $P^{(i)}$ which do not lie in $K^{(t)}+\Lambda^{(i)}$, we have

$$\begin{aligned} m_e(G^{(i)}) &\leqslant \sum_{A\epsilon\Lambda^{(i)}} V\{(R_t+A)\cap P^{(i)}\} \\ &= \sum_{A\epsilon\Lambda^{(i)}} V\{R_t\cap(P^{(i)}+A)\} \\ &= V(R_t\cap R_n) \\ &= R_t, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28) \end{aligned}$$

and

$$m_e(F^{(i)}) = \frac{1}{d(\Lambda^{(i)})} m_e(G^{(i)}) \leqslant \frac{1}{d(\Lambda^{(i)})} R_t < \frac{2}{c} R_t.$$

This proves the lemma.

Now E_i is clearly a subset of $F^{(i)}$. Therefore

$$m_e(E_i) \leqslant m_e(F^{(i)}) < \frac{2}{c} R_t. \qquad \ldots \quad \ldots \quad \ldots \quad (29)$$

From the definition of E_i it is clear that for every $j \geqslant i$, $E_j \supseteq E_i$. Therefore, writing

$$E' = E_1 \cup E_2 \cup \ldots,$$

we have

$$m_e(E) \leqslant m_e(E') = \lim_{i\to\infty} m_e(E_i) \leqslant \frac{2}{c} R_t. \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

If we denote by $H^{(t)}$ the set of points of P the parallelopiped spanned by $Y_1, \ldots, Y_n$ which do not lie in $K^{(t)}+\Lambda$, it follows that

$$m_e(H^{(t)}) = d(\Lambda) m_e(E) \leqslant \frac{2R_t}{c} d(\Lambda) = 2R_t. \qquad \ldots \quad \ldots \quad (31)$$

By taking t large enough we can make R_t as small as we like. This completes the proof of Theorem 4.

We end this section with an example which shows that there exists a K with finite Lebesgue measure for which the possibility that the set of points not lying in $K+\Lambda$ be non-empty does in fact arise.

Example: Let K be the set of points (x, y), where

$$|x| \leqslant \frac{1}{4}, \quad |y| \leqslant \frac{1}{2};$$

$$\frac{1}{4} \leqslant x \leqslant \frac{3}{8}, \frac{1}{2} \leqslant y \leqslant \frac{3}{2}; \quad -\frac{1}{4} \geqslant x \geqslant -\frac{3}{8}, -\frac{1}{2} \geqslant y \geqslant -\frac{3}{2};$$

$$\frac{3}{8} \leqslant x \leqslant \frac{7}{16}, \frac{3}{2} \leqslant y \leqslant \frac{5}{2}; \quad -\frac{3}{8} \geqslant x \geqslant -\frac{7}{16}, -\frac{3}{2} \geqslant y \geqslant -\frac{5}{2};$$

$$\frac{7}{16} \leqslant x \leqslant \frac{15}{32}, \frac{5}{2} \leqslant y \leqslant \frac{7}{2}; \quad -\frac{7}{16} \geqslant x \geqslant -\frac{15}{32}, -\frac{5}{2} \geqslant y \geqslant -\frac{7}{2}; \text{ etc.}$$

It is clear that $c(K) = V(K) = 1$. Also there is no covering lattice Λ, for which $d(\Lambda) = 1$, while the lattice $\Lambda : x = \xi$, $y = \eta$, ξ, η integers has determinant 1 and is such that the set of points which do not lie in $K + \Lambda$ consists of an enumerable number of lines parallel to the Y-axis and is consequently of measure zero.

4. K A STAR-SET.

In this section we shall suppose that K is a star-set,[1] bounded or unbounded. We shall denote its distance function by $f(X)$. As before, we shall write $K^{(t)}$ for the set of points of K which lie in the sphere $|X| \leqslant t$, i.e. $K^{(t)}$ is the set defined by

$$f^{(t)}(X) = \max. \left\{ f(X), \frac{|X|}{t} \right\} \leqslant 1. \qquad \qquad (32)$$

It is well known that $K^{(t)}$ also is a star-set. We shall denote the volume (not necessarily finite) of K by the symbol $V(K)$.

THEOREM 5: *There exist star-sets K with $V(K) = \infty$ for which* (i) $c(K) < \infty$, (ii) $c(K) = \infty$.

Proof: (i) It follows from the result of Davenport referred to in the introduction that K defined by the inequality $|x_1 x_2| \leqslant 1$ has the property (i).

(ii) It is well known that if

$$x_1 = \alpha u + \beta v,$$
$$x_2 = \gamma u + \delta v$$

are linear forms with real coefficients and determinant $\Delta = \left\| \begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix} \right\| \neq 0$ and if α/β is not rational, then for any $\epsilon > 0$ and any real numbers c_1, c_2 there exist integers u, v such that

$$|x_1 + c_1| < \epsilon, \ |(x_1 + c_1)(x_2 + c_2)| < \frac{1}{4} \Delta. \qquad \qquad (33)$$

From (33) it is easy to see that the set $K = f(x) = |(x_1{}^2 x_2)^{\frac{1}{3}}| \leqslant 1$ has arbitrarily large covering lattices. This means that $c(K) = \infty$ and the theorem is proved.

Our next theorem shows that unlike for convex sets with a centre there is no finite upper bound for $\vartheta(K) = V(K)/c(K)$ even when K is restricted to the class of bounded star bodies.

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[1] We use the notation of Mahler (1946).

THEOREM 6:

$$\overline{bd}_{K} \frac{V(K)}{c(K)} = \infty,$$

where the upper bound is taken over all bounded star-sets K.

Proof: Let K be the set $|x_1x_2| \leqslant 1$. Then $K^{(t)}$, the set of its points lying within a distance t from the origin, is a bounded star-set. Since $K^{(t)} \leqslant K$, we have

$$c(K^{(t)}) \leqslant c(K) < 128. \qquad \ldots \quad \ldots \quad \ldots \quad (34)$$

Also $V(K^{(t)}) \to V(K) = \infty$ as $t \to \infty$. Therefore, by taking t large enough we can make $V(K^{(t)})/c(K^{(t)})$ as large as we like and the theorem follows.

Our next theorem is the central theorem for this section.

THEOREM 7: *Let K be an unbounded star body and $K^{(t)}$ the set of its points lying within a distance t from the origin. Then*

$$\lim_{t \to \infty} c(K^{(t)}) = c(K). \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (35)$$

Proof: It is obvious that for $t_2 > t_1$, $K^{(t_2)}$ contains $K^{(t_1)}$ so that $c(K^{(t)})$ is an increasing function of t. If $c(K^{(t)})$ is not bounded above, then there exist real numbers t such that $K^{(t)}$ and so also K has arbitrarily large covering lattices. This means that $c(K) = \infty$ and since $\lim c(K^{(t)}) = \infty$ there is nothing to prove. Therefore, we can suppose that $c(K^{(t)})$ is bounded above so that $c(K^{(t)}) \to$ a limit $c \neq \infty$ as $t \to \infty$ and have to show that $c(K) = c$. For this it will suffice to prove

(*a*) For every $c_1 < c$ there is a covering lattice Λ for K for which $d(\Lambda) > c_1$, and (*b*) there is no covering lattice Λ of K for which we have $d(\Lambda) > c$.

The condition (*a*) is clearly satisfied since for every $c_1 < c$, there is a t with $c(K^{(t)}) > c_1$ and the maximal covering lattice $\Lambda^{(t)}$ of $K^{(t)}$ is a covering lattice for K whose determinant is greater than c_1.

We prove (*b*) by *reductio ad absurdum*. We suppose there is a covering lattice Λ for K, whose determinant $d(\Lambda) = c(1+2\epsilon)^n$, where $\epsilon > 0$ is a real number. The lattice $\Lambda' = \frac{1}{1+\epsilon}\Lambda$ is obviously a covering lattice for the set $\frac{1}{1+\epsilon}K$. Let $X_1, \ldots, X_n$ be a basis of Λ' and let P be the corresponding fundamental cell. Let $X = \alpha_1X_1 + \ldots + \alpha_nX_n$ where $0 \leqslant \alpha_i \leqslant 1$ for all $i = 1, \ldots, n$ be a point of P. Then there exists a point $A(X) = A$ of Λ' such that

$$f(X-A) \leqslant \frac{1}{1+\epsilon}.$$

Since $f(X)$ is continuous we can find an open sphere $C(X)$ with centre at X such that for every point Z of $C(X)$ we have

$$f(Z-A) \leqslant 1.$$

In this way we can associate an open sphere $C(X)$ and a point $A(X)$ of Λ' with every point X or P such that all points in $C(X)$ lie in the set $K+A(X)$. As P is a closed bounded set, we can choose a finite number of spheres $C(X)$ which together cover P. This means that we can choose a finite number of points $A_1, \ldots, A_\mu$ of Λ' such that P is contained in the union of the sets $K+A_1, \ldots, K+A_\mu$. As μ is finite there exists a t such that P is completely contained in the union of the sets $K^{(t)}+A_1, \ldots, K^{(t)}+A_\mu$. This means that Λ' is a covering lattice for $K^{(t)}$. Since $d(\Lambda') = \frac{1}{(1+\epsilon)^n} d(\Lambda) > c \geqslant c(K^{(t)})$ we get a contradiction and the theorem is completely proved.

We next give three theorems which are consequences of Theorem 7 and are analogous to certain theorems of Mahler (1946) and Davenport and Rogers (1949) for critical determinants of star bodies.

THEOREM 8: *Let K, K_1, K_2, ... be an infinity of star bodies satisfying the following conditions :—*

(*a*) *to every $\epsilon > 0$ there corresponds a positive integer $N(\epsilon)$ such that for all $r \geqslant N(\epsilon)$, K_r is contained in $(1+\epsilon)K$.*

(*b*) *for every $t > 0$ and every $\epsilon > 0$, there exists a positive integer $N(t, \epsilon)$ such that $K^{(t)}$ is contained in $(1+\epsilon)K_r$ for all $r \geqslant N(t, \epsilon)$. Then*

$$\lim_{r \to \infty} c(K_r) = c(K). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (36)$$

Proof: For r large enough K_r is contained in $(1+\epsilon)K$,

therefore $$c(K_r) \leqslant c((1+\epsilon)K_r) = (1+\epsilon)^n c(K).$$

Making $\epsilon \to 0$, we have

$$\limsup_{r \to \infty} c(K_r) \leqslant c(K).$$

Again for r large enough $K^{(t)}$ is contained in $(1+\epsilon)K_r$.

Therefore $$c(K^{(t)}) \leqslant c((1+\epsilon)K_r) = (1+\epsilon)^n c(K_r).$$

Therefore, on making $\epsilon \to 0$ and $t \to \infty$, we get

$$c(K) \leqslant \liminf_{r \to \infty} c(K_r),$$

and the theorem follows.

THEOREM 9: *Let $K : f(X) \leqslant 1$ be a star-set, $g(X)$ an arbitrary distance function and t a positive parameter. Then for the star-sets $K_t(X) : f_t(X) \leqslant 1$, where*

$$f_t(X) = \max. \{ f(X), t^{-1}g(X) \}, \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

we have

$$\lim_{t \to \infty} c(K_t) = c(K). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (38)$$

Proof: Since $f_t(X) \geqslant f(X)$ for all X and t, it is clear that K_t is contained in K. Also since the set $H : g(X) \leqslant 1$ is a star-set, there exists a number $\tau > 0$ such that H contains the sphere $|X| \leqslant \tau$. The sphere $|X| \leqslant \tau t$ is contained in tH, so that $K^{(\tau t)}$ which is a subset of this sphere lies in tH. Consequently $K^{(\tau t)}$ is contained in K_t. Thus the hypotheses of Theorem 8 are satisfied and (38) follows.

THEOREM 10: *Let $g(x_1, \ldots, x_r)$ and $e(x_1, \ldots, x_r)$ be distance functions of r variables and let $h(x_{r+1}, \ldots, x_n)$ be a distance function of $n-r$ variables, where $1 \leqslant r \leqslant n-1$. Then the star bodies in R_n defined by*

$$\{g(x_1, \ldots, x_r)\}^{r/n}\{h(x_{r+1}, \ldots, x_n)\}^{\frac{n-r}{n}} \leqslant 1 \quad \ldots \quad \ldots \quad (39)$$

and

$$\{g(x_1, \ldots, x_r)\}^{r/n}\{e(x_1, \ldots, x_r)+h(x_{r+1}, \ldots, x_n)\}^{\frac{n-r}{n}} \leqslant 1 \quad \ldots \quad (40)$$

have the same covering constant.

Proof: Let K denote the star-set defined by (39) and K' that defined by (40). The substitution

$$\nu x_1 = x_1', \ldots, \nu x_r = x_r', \mu x_{r+1} = x_{r+1}', \ldots, \mu x_n = x_n',$$

where ν and μ are positive numbers with $\nu^r \mu^{n-r} = 1$ as a substitution of determinant 1. Hence, if K_ν' denotes the star body defined by

$$\left\{g\left(\frac{x_1}{\nu}, \ldots, \frac{x_r}{\nu}\right)^{r/n}\right\}\left\{e\left(\frac{x_1}{\nu}, \ldots, \frac{x_r}{\nu}\right)+h\left(\frac{x_{r+1}}{\mu}, \ldots, \frac{x_n}{\mu}\right)\right\}^{\frac{n-r}{n}} \leqslant 1 \quad .. \quad (41)$$

we have

$$c(K_\nu') = c(K') \quad .. \quad .. \quad .. \quad .. \quad (42)$$

for any ν.

Since g, e and h are distance functions we can write (41) as

$$g(x_1, \ldots, x_r)^{r/n}\left\{\frac{\mu}{\nu} e(x_1, \ldots, x_r)+h(x_{r+1}, \ldots, x_n)\right\}^{\frac{n-r}{n}} \leqslant 1. \quad .. \quad (43)$$

Now consider any point $(x_1, \ldots, x_n)$ of $K^{(t)}$. This point satisfies (39) and g, e, h have upper bounds g_0, e_0, h_0 which depend on t but not on ν. We, therefore, have

$$g^{r/n}\left(\frac{\mu}{\nu} e+h\right)^{\frac{n-r}{n}} \leqslant g^{r/n}h^{\frac{n-r}{n}}+g^{r/n}\left(\frac{\mu}{\nu} e\right)^{\frac{n-r}{n}}$$

$$\leqslant 1+g_0^{r/n}\left(\frac{\mu}{\nu} e_0\right)^{\frac{n-r}{n}}.$$

If ϵ is any positive number, the last expression can be made smaller than $1+\epsilon$ by taking ν large enough. Hence for a fixed t the body $K^{(t)}$ is contained in $(1+\epsilon)K_\gamma'$ for large ν. Therefore

$$c(K^{(t)}) \leqslant (1+\epsilon)^n c(K_\nu') = (1+\epsilon)^n c(K'),$$

so that

$$c(K^{(t)}) \leqslant c(K'),$$

and by making $t \to \infty$

$$c(K) \leqslant c(K'). \quad .. \quad .. \quad .. \quad .. \quad (44)$$

But K' is contained in K. Therefore

$$c(K') \leqslant c(K), \quad .. \quad .. \quad .. \quad .. \quad (45)$$

and the theorem follows from (44) and (45).

It is easy to verify that the theorem remains valid if in (40) $e+h$ is replaced by $(e^\alpha+h^\alpha)^{1/\alpha}$ where $\alpha > 1$.

As a consequence of this theorem it is interesting to note that for any positive k, the sets

$$K_1 : |xy| \leqslant 1, \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (46)$$

$$K_2 : |x|(k|x|+|y|) \leqslant 1, \quad .. \quad .. \quad .. \quad .. \quad (47)$$

and

$$K_3 : |x|(k|x|^\alpha+|y|^\alpha)^{1/\alpha} \leqslant 1, \alpha > 1, \quad .. \quad .. \quad (48)$$

have the same covering constant. Further, as the set $K_4 = |x| \leqslant \frac{1}{\sqrt{k}}, |xy| \leqslant 1$

is contained in K_1, while K_2 is contained in K_4, it is interesting to see that the set K_1 has the same covering constant as its subsets $|xy| \leqslant 1$, $|x| \leqslant \epsilon$, for all ϵ, however small.

Summary.

Certain general theorems about the covering constant and covering lattices have been obtained for different types of sets in the Euclidean space of n dimensions.

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