

ON POLAR RECIPROCAL CONVEX DOMAINS

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1. Let k and K be two symmetrical convex domains in the plane with centres at the origin. Suppose k and K are polar reciprocal with respect to the unit circle C centred at the origin.

Let $\Delta(K)$ be the critical determinant* of K and $c(k)$ the covering constant of k , i.e. the upper bound of the determinants of the covering lattices for k , where a lattice A is called a covering lattice for k if every point in the plane lies in one at least of the bodies obtained from k by applying to it all possible lattice translations.

Mahler (1948) proved that

$$\frac{1}{2} \leq \Delta(K) \Delta(k) \leq \frac{3}{4}, \quad \dots \quad \dots \quad \dots \quad \dots \quad (A)$$

and that both the inequalities are best possible.

It seems worth noticing that

$$2 \leq \Delta(K) c(k) \leq \frac{9}{4}, \quad \dots \quad \dots \quad \dots \quad \dots \quad (1)$$

and that both inequalities here again are best possible.

2. In this section we prove (1).

Write $H_s(K)$ for the area of the smallest symmetrical convex hexagon† circumscribed to K and $h_s(k)$ for the area of the largest symmetrical convex hexagon† inscribed in k . Then it is known that

$$\Delta(K) = \frac{1}{4} H_s(K), \quad \dots \quad \dots \quad \dots \quad \dots \quad (2)$$

and

$$c(k) = h_s(k); \quad \dots \quad \dots \quad \dots \quad \dots \quad (3)$$

(2) follows from Theorem 1 of Mahler (1948) and Theorem 3 of Dowker (1944) while (3) is lemma 8 of Bambah and Rogers (1952).

Let H be the symmetrical convex hexagon circumscribed to K with area $a(H) = H_s(K)$. Then H' , the polar reciprocal of H , is a symmetrical convex hexagon inscribed in k so that

$$c(k) = h_s(k) \geq a(H'),$$

where $a(H')$ is the area of H' . Consequently

$$\Delta(K) c(k) = \frac{1}{4} H_s(K) h_s(k) \geq \frac{1}{4} a(H) a(H') \geq 2; \quad \dots \quad \dots \quad (4)$$

the last inequality following from inequalities A of Mahler (1948).

* For definition see e.g., K. Mahler, *Proc. Royal Soc., A*, 187 (1946), 151.

† By a hexagon we mean a polygon with at most six sides.

Now suppose h is the convex symmetrical hexagon of area $a(h) = h_s(k)$ inscribed in k . Then h' , its polar reciprocal, is a convex symmetrical hexagon circumscribed to K , so that

$$\Delta(K) = \frac{1}{4}H_s(K) \leq \frac{1}{4}a(h'),$$

where $a(h')$ is the area of h' . Therefore we have

$$\Delta(K) c(k) = \frac{1}{4}H_s(K) h_s(k) \leq \frac{1}{4}a(h') a(h) \leq \frac{9}{4}, \quad \dots \dots (5)$$

the last inequality following from inequalities *A* of Mahler (1948).

(4) and (5) together prove (1).

Let k and K be the squares:

$$k: |x| \leq 1; |y| \leq 1; K: |x| + |y| \leq 1.$$

Then

$$c(k) = 4, \Delta(K) = \frac{1}{2} \text{ and } c(k) \Delta(K) = 2. \dots \dots (6)$$

If k and K become the unit circle or the regular hexagon inscribed in the unit circle and its polar reciprocal, then

$$c(k) = \frac{3\sqrt{3}}{2}, \Delta(K) = \frac{\sqrt{3}}{2},$$

and

$$c(k) \Delta(K) = \frac{9}{4}, \quad \dots \dots (7)$$

which together with (6) shows that the inequalities (1) are best possible.

3. From the well-known results

$$3^{-n+1} V(K) \leq c(K) \leq V(K) \text{ (Rogers),}$$

$$2^{-n} V(k) \leq \Delta(k) \leq \frac{1}{a_n} V(k), \quad a_n \rightarrow 4.921 \text{ as } n \rightarrow \infty$$

(Minkowski and Davenport-Rogers)

and

$$\frac{2^n J_n}{(n! n^n)^{\frac{1}{2}}} \leq V(k) V(K) \leq J_n^2$$

(Dvoretzky Rogers and Santalo)

(where J_n is the volume of the unit sphere in n dimensions), one can find analogous inequalities for $c(k) \Delta(K)$ and $c(k) c(K)$ for polar reciprocal symmetrical convex bodies k and K in n dimensions but they appear to be far from best possible.

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