A GENERALIZATION OF CANTOR BENDIXON THEOREM

by K. Padmavally, *Research Fellow, N.I.S.I., Ramanujan Institute of Mathematics, Madras*

(Communicated by T. Vijayaraghavan, F.N.I.)

(*Received January 6 ; read May 7, 1954*)

Cantor Bendixon theorem gives an upper limit to the stage at which the process of repeated derivation* of a set in Euclidean space stops. The final form of the result given by Cantor (1884) is: For every subset X of a Euclidean space there exists an ordinal $\gamma\ [=\gamma(X)]$ of the first or second class such that $X^{(\gamma)}$, the γ-th derivative of X, $= X^{(\gamma+1)}$. This note gives a partial extension of this theorem for spaces other than Euclidean spaces. The result is based on the notion of 'intersection character' of a space at a point, which is defined by Alexandroff (1924) as the least cardinal number of a family of neighbourhoods of the point whose intersection is the point. The intersection character of a T_1-space is evidently defined at every point. Also for a Euclidean space the intersection character at every point is $\aleph_0$, so that $\aleph_1$ is a regular† cardinal number greater than the intersection character of the space at every point. Cantor Bendixon theorem implies that for every subset X of a Euclidean space, $X^{(\omega_1)} = X^{(\omega_1+1)}$ (ω_1 being the initial ordinal corresponding to the cardinal $\aleph_1$, i.e., the first ordinal of the class of ordinals each having power $\aleph_1$). The following result will be seen to be a generalization of this statement.

If R is a regular space (i.e. a T_1-space in which every neighbourhood of any point contains the closure of another neighbourhood of the point) *which is locally bicompact at every point and $\aleph_\alpha$ any regular cardinal number greater than the intersection character of R at every point, then for every set $X \subset R$, $X^{(\omega_\alpha)} = X^{(\omega_\alpha+1)}$.*

If possible let p be a point of $X^{(\omega_\alpha)} - X^{(\omega_\alpha+1)}$. Then p is an isolated point of $X^{(\omega_\alpha)}$ and hence has a neighbourhood U such that $U \cap X^{(\omega_\alpha)} = p$. Since R is regular, $U \supset \overline{V}$, the closure of another neighbourhood V of p. Also since R is locally bicompact at p, p has a neighbourhood W such that $\overline{W}$ is bicompact. Let $\aleph_p$ denote the intersection character of R at p. Then p has a set $\{U_\gamma\}_{\gamma<\omega_p}$ of neighbourhoods such that $\prod_{\gamma<\omega_p} U_\gamma = p$. Let $V_\gamma = U_\gamma \cap V \cap W$. Then $\overline{V}_0 \cap X^{(\omega_\alpha)} = p$ and $\prod_{\gamma<\omega_p} V_\gamma = p$. Hence for each of the sets $\overline{V}_0 - V_\gamma$ (which is closed since V_γ can be taken to be open), $(\overline{V}_0 - V_\gamma) \cap X^{(\omega_\alpha)} = 0$, and $\overline{V}_0 = p \cup \sum_{\gamma<\omega_p} (\overline{V}_0 - V_\gamma)$. Also $X^{(\omega_\alpha)} \cap (\overline{V}_0 - V_\gamma)$ is the inner limiting set of the decreasing series of closed sub-sets $\{(\overline{V}_0 - V_\gamma) \cap X^{(\beta)}\}$, $\beta < \omega_\alpha$, of the bicompact space $\overline{W}$.

* For any ordinal β the β-th derivative $X^{(\beta)}$ of X is defined by induction as $= \prod_{\gamma<\beta} X^{(\gamma)}$ if β is a limiting ordinal, and = the derivative of $X^{(\gamma)}$ if β is a nonlimiting ordinal $= \gamma+1$.

† The initial ordinal ω_α of the class of ordinals each having power $\aleph_\alpha$ is said to be *regular* (Hausdorff, 1908) if every subseries of ω_α cofinal with ω_α is similar to ω_α.

Therefore some member of the series is null, i.e. for each $\gamma < \omega_p$ there exists $\beta\ [=\beta\ (\gamma)] < \omega_\alpha$ such that $(\bar{V}_0 - V_\gamma) \cap X^{\beta\ (\gamma)} = 0$. But since by hypothesis the cardinal number $\aleph_p$ of the set $\{\beta\ (\gamma)\}_{\gamma<\omega_p}$ is less than $\aleph_\alpha$, and each $\beta\ (\gamma) < \omega_\alpha$, and ω_α is a regular initial ordinal, there exists $\beta' < \omega_\alpha$ such that each $\beta(\gamma) \leq \beta'$. Hence each $(\bar{V}_0 - V_\gamma) \cap X^{(\beta')} = 0$, and therefore $X^{(\beta')} \cap \bar{V}_0 = \{p \cup \Sigma_{\gamma<\omega_p} (\bar{V}_0 - V_\gamma)\} \cap X^{(\beta')} \subset p$. Hence p is an isolated point of $X^{(\beta')}$ and therefore p is not $\epsilon\ X^{(\beta'+1)}$, which contradicts the hypothesis that $p \ \epsilon\ X^{(\omega_\alpha)}$, since $\beta'+1 < \omega_\alpha$ (β' being $< \omega_\alpha$ and ω_α being a limiting ordinal).

Since a Euclidean space is regular and locally bicompact at all points, this clearly implies the result stated above. The result corresponding to Cantor Bendixon theorem for a T_1-space will be: If $\aleph_\alpha$ is a regular cardinal number greater than the intersection character of the space at every point, then for every set X there exists an ordinal $\gamma\ [=\gamma\ (X)] < \omega_\alpha$ such that $X^{(\gamma)} = X^{(\gamma+1)}$. The following example shows that this result does not hold in general for all regular bicompact spaces.

Consider the ordered set R consisting of all complexes (x_0, x_1) where x_0, x_1 are any numbers in the interval $0 \leq x \leq 1$, ordered so that $(x_0{}^1, x_1{}^1) < (x_0{}^2, x_1{}^2)$ if either $x_0{}^1 < x_0{}^2$, or $x_0{}^1 = x_0{}^2$ and $x_1{}^1 < x_1{}^2$, under its order topology (in which the open sets are all open intervals). The intersection character of every point is clearly $\aleph_0$, since every point is easily seen to be the limit of a well ordered sequence similar to ω as well as an inversely well ordered sequence similar to ω^*, the inverse of ω. But the process of derivation of the set X constructed below can be seen to stop only at the ω_1-th stage.

Since the set of values of x_0 (which is the set of all numbers in (0, 1)) has power $2^{\aleph_0} \geqslant \aleph_1$, it is possible to find a subset of this set say, Q, having power $\aleph_1$, which can therefore be well ordered as an ω_1 series $\{x_0{}^\beta\}_{\beta<\omega_1}$ ($x_0{}^\beta$ being $= x_0{}^{\beta'}$ if and only if $\beta = \beta'$). For each $\beta < \omega_1$ let I_β denote the interval or R given by (i.e., consisting of all complexes satisfying the equation) $x_0 = x_0{}^\beta$ (= constant), (x_1 being unrestricted in the open interval (0, 1)). Then the I_β's form a set of disjoint intervals each similar to the open real number interval (0, 1). It has been proved by Cantor that in any interval of the real number space corresponding to any given ordinal γ a set X can be found such that $X^{(\beta)} \neq X^{(\beta+1)}$ for all $\beta < \gamma$. Hence in each interval I_β a set X_β can be found such that $X_\beta^{(\gamma)} \neq X_\beta^{(\gamma+1)}$ for all $\gamma < \beta$. Then if $X = \bigcup_{\beta<\omega_1} X_\beta$, it is clear that since the intervals I_β are disjoint with each other, for every $\gamma < \omega_1$, $X^{(\gamma)} \cap I_\beta = X_\beta^{(\gamma)}$ and hence for every $\gamma < \omega_1$ $X^{(\gamma)} \neq X^{(\gamma+1)}$.

This paper formed part of a thesis submitted for the M.Sc. Degree of the Madras University. I wish to thank Dr. R. Vaidyanathaswamy, Dr. M. Venkataraman and Dr. V. S. Krishnan for their help and guidance.

Summary

This paper gives a generalization of a well-known result of Cantor and Bendixon to Hausdorff bicompact spaces.

Bibliography

Alexandroff (1924). Über die struktur der bikompakten topologischen Räume. *Math. Ann.*, **92**, 267–274.

Cantor (1884). Ueber unendliche Punktmannichfaltigkeiten, *Ibid.*, **23**, 453–488.

Hausdorff (1908). Grundzuge einer Theorie der geordneten Mengen, *Ibid.*, **65**, 435–505.

Issued June 26, 1954.