ON POLAR RECIPROCAL CONVEX DOMAINS

Addendum

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1. Let k and K be two plane symmetrical convex domains centred at the origin O, that are polar reciprocal to each other with respect to the unit circle centred at O. In a recent note* [1954] the author proved the best possible inequalities

$$2 \leqq \Delta\,(k)\, c(K) \leqq \tfrac{9}{4},$$

The object of this addendum is to add the companion inequalities

$$27/4 \leqq c(k)\, c(K) \leqq 9\,, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

which are also best possible.

Throughout this note we use the notation of PRCD.

2. If H is a symmetrical convex hexagon with vertices $P_1, P_2, P_3, -P_1, -P_2, -P_3$, occurring in that order, then the triangle $P_1P_3{-}P_2$ has area $a(P_1P_3{-}P_2) = \frac{1}{2}\,a(H)$. Also if T is a triangle with area $a(T)$, then the hexagon formed by taking the convex cover of T and $-T$ (the image of T in O) has area at least $2a(T)$. From these observations one easily concludes that in the notation of PRCD

$$c(k) = h_s(k) = 2t(k),\ c(K) = h_s(K) = 2t(K), \qquad \ldots \quad \ldots \quad (2)$$

where $t(k)$ and $t(K)$ denote the areas of the biggest triangles inscribed in k and K respectively.

We next prove the following lemma, which is a particular case of one due to Mahler, valid in n dimensions:

Lemma : *If T is a triangle containing O and T' is its polar reciprocal, then*

$$a(T)\, a(T') \geqq 27/4.$$

Let the vertices P, Q, R of T occur in the counter-clockwise order and let their co-ordinates be (x_1, y_1), (x_2, y_2), and (x_3, y_3) respectively. Since O lies inside T, there exist positive numbers ξ, η, ζ such that

$$x_1\xi + x_2\eta + x_3\zeta = 0\,,$$

$$y_1\xi + y_2\eta + y_3\zeta = 0\,,$$

so that, for some real number ρ, we have

$$\xi = \rho(x_2y_3 - x_3y_2),\ \eta = \rho(x_3y_1 - x_1y_3),\ \text{and}\ \zeta = \rho(x_1y_2 - x_2y_1).$$

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* We refer to this Note as PRCD.

By straight-forward calculations one easily obtains

$$a(T) = \frac{1}{2\rho}(\xi+\eta+\zeta),$$

and

$$a(T') = \tfrac{1}{2}\rho(\xi+\eta+\zeta)^2/\xi\eta\zeta$$

so that

$$a(T)\,a(T') = \tfrac{1}{4}(\xi+\eta+\zeta)^3/\xi\eta\zeta \geqq 27/4.$$

Now suppose t is a triangle inscribed in K with area $a(t) = t(K)$. Then, since t is maximal and K is convex it is easy to see that the lines through the vertices of t parallel to opposite sides are tac lines of K, so that K is contained in T, the triangle formed by these lines. Clearly

$$a(T) = 4a(t) = 4t(K).$$

Since K contains O in the interior, so also does T. Consequently if t' denotes the polar reciprocal of T, then

$$a(T)\,a(t') \geqq 27/4.$$

Also t' lies in k. Therefore

$$c(k) = 2t(k) \geqq 2a(t'),$$

and

$$c(K)\,c(k) = 4t(K)\,t(k) \geqq 4a(t)\,a(t') = a(T)\,a(t') \geqq 27/4. \quad \ldots \quad \ldots \quad (3)$$

Next suppose H is a hexagon inscribed in K with $a(H) = h_s(K)$. Then H', its polar reciprocal, contains k, so that

$$c(k) \leqq a(k) \leqq a(H'),$$

and

$$c(K)\,c(k) \leqq a(H)\,a(H') \leqq 9, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

the last inequality follows from inequalities A of Mahler [1948]. Inequalities (3) and (4) prove (1).

If k is a regular hexagon inscribed in the unit circle and K its polar reciprocal then

$$c(k) = \frac{3\surd 3}{2},\ c(K) = 2\surd 3, \text{ and } c(k)\,c(K) = 9. \quad \ldots \quad \ldots \quad (5)$$

On the other hand, if k, and so also K, is the unit circle, then

$$c(k) = \frac{3\surd 3}{2},\ c(K) = \frac{3\surd 3}{2}, \text{ and } c(k)\,c(K) = 27/4,$$

which together with (5) shows that inequalities (1) are best possible.

References.

Bambah, R. P. (1954). On Polar reciprocal Convex Domains. *Proc. Nat. Inst. Sci.*, **20**, 119-120.
Mahler, K. (1948). *Proc. Kon. Ned. Akad. V. Weten., Amsterdam*, **51**, 482–485.

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