INSTABILITY OF NON-RADIAL OSCILLATIONS OF CENTRALLY CONDENSED STARS

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1. Introduction

Generally in theoretical discussions of stellar pulsation, non-radial modes of oscillations are left out, with the justification that these modes are likely to be associated with a greater amount of viscous damping than in purely radial oscillations. But it is plausible to argue, that in the lowest modes of non-radial oscillations the viscous damping is not likely to exceed by a great deal the damping in radial oscillations. Also we must take into account the circumstance that, whatever external disturbances may exist, they will always be of a non-radial character. A point of great consequence is, as pointed out by Rosseland, that a limitation to purely radial oscillations fails to reveal the possible instability of the assumed model for a more general type of disturbance. If there is any mode of oscillation for which the assumed model is unstable, then the possibility of the existence of a star built on that model is nil; and therefore the periods of purely radial oscillations of the model can have no application to the actual stars. An investigation of non-radial oscillations should thus lead to an elimination of a large class of oscillating stellar models.

The Eulerian equations which govern the small adiabatic oscillations of non-rotating stars were first derived by Rosseland (1932) in connection with his investigation of the effect of non-adiabatic processes on the stability of stars. The oscillations are found to be governed by an ordinary differential equation of the fourth order, the explicit form of which was eventually obtained by Pekeris (1938). Nothing is known, however, about the solutions of this equation beyond the simplest case of an initially homogeneous sphere, for which an instability with respect to all non-radial oscillations was established by Pekeris.

In order to simplify the problem and gain some insight into the nature of its characteristic frequency spectrum, Emden (1907) conceived the idea of ignoring the effect of the displacement upon the gravitational potential in the outer parts of a centrally condensed configuration—a simplification which reduces the order of the governing differential equation from fourth to second. His work was, however, vitiated by inconsistent approximations made in considering the equation of continuity which he reduced in fact to the form appropriate to a homogeneous incompressible fluid. The correct formulation of a problem so simplified was later given by Cowling (1942). We shall utilise Emden's device in deducing the equations of the problem.

2. Equations of Motion.

Consider the oscillations of a gas sphere under the influence of its own gravity, the oscillations being so small that squares of amplitudes of the displacements and their derivatives can be ignored in comparison with their first powers. In setting up the equations of motions, we shall neglect viscous forces and assume that the motion of any individual gas particle takes place adiabatically.

Let the pressure P, density ρ, and gravitational potential V at any point be altered by δP, $\delta\rho$ and δV respectively, and let the vector displacement of material from its equilibrium position be $\hat{h}$. Then written within the scope of our approximations, the Eulerian equations of motion can be written as follows:—

$$P\frac{\partial^2 \hat{h}}{\partial t^2} + \text{grad}\,\delta P + \delta P\,\text{grad}\,V + \rho\,\text{grad}\,\delta V = 0 \quad \ldots \quad \ldots \quad (1)$$

where t denotes the time, and the equation of continuity becomes

$$\delta\rho + \text{div}\,(\rho\hat{h}) = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Assuming the motion to be simple harmonic we shall seek such solutions of the foregoing system of equations for which

$$\frac{\partial^2 \hat{h}}{\partial t^2} + \sigma^2 \hat{h} = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where $\sigma = \frac{2\pi}{\tau}$, τ being the period of the respective oscillation.

Equations (1) and (2) represent a simultaneous system of the fourth order, the solution of which presents, in any case, a problem of great mathematical complexity. In order to simplify it to some extent we follow Emden (1907), Rosseland (1932) and Cowling (1942) and ignore the variation of the gravitational potential on the left side of equation (1). Since in the Roche Model nearly the whole mass is concentrated at its centre, the variation in density accompanying the oscillations will produce but minor variations in the gravitational potential through the interior of the star. The neglect of the variation in gravitational potential will not, therefore, affect in any appreciable manner the characteristic features of our problem. On the other hand, the advantage of such a neglect is the reduction of our mathematical problem to the solution of a single differential equation of the second order.

The fundamental equations are ultimately reducible to the following as in Kopal (1949) :—

$$\frac{d\zeta}{dr} - \frac{g}{c^2}\zeta = \left\{\frac{j(j+1)}{\sigma^2} - \frac{r^2}{c^2}\right\}\cdot\frac{\eta}{\rho} \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

$$\frac{d\eta}{dr} + \frac{g}{c^2}\cdot\eta = \left\{\sigma^2 + \frac{g}{\rho}\cdot\frac{d\rho}{dr} + \frac{g^2}{c^2}\right\}\cdot\frac{\rho}{r^2}\cdot\zeta \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

where

$\zeta = r^2 ar$

$\eta = \delta P$

$g =$ gravity

$c^2 = \gamma\,\frac{P}{\rho}$ where $\gamma =$ ratio of specific heats

$j =$ order of surface harmonic

$\sigma =$ frequency of oscillation.

The boundary conditions of the problem call for a node (no displacement) at the centre and a loop (no variation in pressure) at the free surface of the oscillating configuration, i.e. they require that $\delta r = 0$ at $r = 0$, while $\delta P = 0$ at a value of r_1, for which $\rho(r_1) = 0$. These conditions can be satisfied only for particular values of σ.

We first eliminate η between (4) and (5). Put

$$\rho = r^2 dr$$
$$= r^2\xi$$

where ξ is the amplitude of oscillations and $r = Rx$ where R is the radius of the star. Elimination of ζ gives

$$x^2\frac{d^2\xi}{dx^2}+\left[4x-Rx^2\frac{A'}{A}\right]\frac{d\xi}{dx}+\left[2-2Rx\frac{A'}{A}-Rx^2A\frac{d}{dx}(g/Ac^2)-\frac{g^2R^2x^2}{c^4}-x^2R^2AB\right]\xi = 0 \quad \ldots \quad (6)$$

where

$$A = \left[\frac{j(j+1)}{\sigma^2}-\frac{R^2x^2}{c^2}\right]\cdot\frac{1}{\rho}$$

$$= \left[k-\frac{R^2x^2}{c^2}\right]\cdot\frac{1}{\rho} \text{ where } k = \frac{j(j+1)}{\sigma^2} \quad \ldots \quad \ldots \quad (7)$$

$$B = \left[\sigma^2+\frac{g}{\rho}\cdot\frac{d\rho}{dr}+\frac{g^2}{c^2}\right]\frac{\rho}{R^2x^2} \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

$$A' = \frac{dA}{dr}$$

We now apply equation (6) to the Roche model. Let M denote practically the whole mass of the star which is concentrated at its centre.

Then
$$g = \frac{GM}{R^2x^2} = \frac{\mu}{R^2x^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

where G is the gravitational constant.

$$\rho = \rho_0\, x^{-2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

Therefore
$$p = \int_r^R \rho_0 x^{-2}\frac{GM}{R^2}x^{-2}\,d(Rx)$$

$$= \frac{GM\rho_0}{3Rx^3}(1-x^3) \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

Hence
$$c^2 = \frac{\gamma p}{\rho}$$

$$= \frac{\gamma GM}{3Rx}(1-x^3) = \frac{\alpha(1-x^3)}{Rx} \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

where
$$\alpha = \frac{\gamma GM}{3}$$

Substitution of these values reduces (6) after simplification to

$$P\frac{d^2\xi}{dx^2}+Q.\frac{d\xi}{dx}+S.\xi = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

where

$$\left.\begin{aligned} P &= p_5x^5+p_8x^8+p_{11}x^{11}+p_{14}x^{14} \\ Q &= q_4x^4+q_7x^7+q_{10}x^{10}+q_{13}x^{13} \\ S &= s_0+s_3x^3+s_6x^6+s_9x^9+s_{12}x^{12} \end{aligned}\right\} \quad \ldots \quad (14)$$

where

$$\left.\begin{aligned} p_5 &= R^3k\alpha \\ p_8 &= -3R^3k\alpha-R^6 \\ p_{11} &= 3R^3k\alpha+2R^6 \\ p_{14} &= -R^3k\alpha-R^6 \\ q_4 &= 2\alpha kR^3 \\ q_7 &= -6\alpha kR^3+R^6 \\ q_{10} &= 6\alpha kR^3+R^6 \\ q_{13} &= -2\alpha kR^3-2R^6 \\ s_0 &= 2\mu\alpha k^2-k^2\mu^2 \\ s_3 &= R^3\left(\frac{k\mu^2}{\alpha}-k^2\sigma^2\alpha-\mu k-2\alpha k\right) \\ &\quad -6\mu k^2\alpha+2k^2\mu^2 \\ s_6 &= 6\mu k^2\alpha-k^2\mu^2+R^3\left(6k\alpha-\mu k+3k^2\alpha\sigma^2-\frac{k\mu^2}{\alpha}\right) \\ &\quad +R^6\left(8-\frac{4\mu}{\alpha}+2k\sigma^2\right) \\ s_9 &= -2\mu k^2\alpha+R^3(-6k\alpha+2\mu k-3k^2\alpha\sigma^2) \\ &\quad +R^6\left(-10+\frac{4\mu}{\alpha}-4k\sigma^2\right)+R^9\left(-\frac{\sigma^2}{\alpha}\right) \\ s_{12} &= R^3(2k\alpha+k^2\alpha\sigma^2)+R^6(2+2k\sigma^2)+\frac{\sigma^2}{\alpha}\cdot R^9 \end{aligned}\right\} \quad \ldots \quad (15)$$

3. Solution of the differential equation

The equation to be solved is

$$P\frac{d^2\xi}{dx^2}+Q\frac{d\xi}{dx}+S\xi = 0$$

P can be put in the form

$$-R^3x^4(x^3-1)^2(\beta^3x^3-1) \text{ where } \beta^3 = 1+\frac{R^3}{k\alpha}$$

Q can be put in the form

$$-R^3x^5(x^3-1)[2\alpha k(1-x^3)^2+R^3x^3(1+2x^3)]$$

S can be put in the form

$$\begin{aligned} -(x^3-1)\Big[&\{2\mu\alpha k^2-k^2\mu^2\} \\ &-x^3\left\{4\mu k^2\alpha-k^2\mu^2-\frac{R^3k\mu^2}{\alpha}+k^2\sigma^2\alpha R^3+\mu kR^3+2\alpha kR^3\right\} \\ &+x^6\left\{2\mu k^2\alpha+4k\alpha R^3-2\mu kR^3+2k^2\sigma^2\alpha R^3+8R^6-\frac{4\mu}{\alpha}\cdot R^6+2k\sigma^2R^6\right\} \\ &+x^9\left\{-2k\sigma^2R^6-2R^6-k^2\alpha\sigma^2R^3-\frac{\sigma^2}{\alpha}R^9-2k\alpha R^3\right\}\Big] \end{aligned}$$

On cancelling throughout $-(x^3-1)$, it is easily seen that there are three singularities

(i) at $x = 0$,
(ii) at $x = 1$,
and (iii) at $x = \frac{1}{\beta}$, i.e. in between the centre and the surface of the star.

Also $x = 0$ is an irregular singularity and therefore a series solution in ascending positive powers is not possible. Further, due to the singularity in between the centre and the surface of the star the problem seems intractable in its present form.

4. Solution of the Problem in the case for which $\gamma = 1{\cdot}5$

We, therefore, consider a restricted problem. We seek a solution for the case in which the constant term in S disappears and therefore the origin becomes a regular singularity.

This condition gives :—

$$2\mu\alpha k^2 - k^2\mu^2 = 0$$

or
$$2\alpha = \mu$$

or
$$2\gamma\frac{GM}{3} = GM$$

or
$$\gamma = 1{\cdot}5$$

With this value of γ, the differential equation reduces to

$$P'\xi'' + Q'\xi' + S'\xi = 0 \qquad \ldots \qquad (16)$$

where

$$P' = R^3x^2(x^3-1)(\beta^3x^3-1)$$
$$Q' = R^3x[2\alpha k(1-x^3)^2 + R^3x^3(1+2x^3)]$$
$$S' = [-(k^2\sigma^2\alpha R^3 + 4k^2\alpha^2) + x^3(4k^2\alpha^2 + 2R^3k^2\alpha\sigma^2 + R^6 2k\sigma^2) - x^6(2R^3k\alpha + k^2\alpha\sigma^2R^3 + 2R^6 + 2R^6k\sigma^2)]$$

The regular singularities are $x = 0$, $x = 1$, $x = \frac{1}{\beta}$.

Since $\beta = 1 + \frac{R^3}{k\alpha}$, $\frac{1}{\beta} < 1$, therefore there is a singularity within the star between the centre and the surface.

We first determine the nature of this singularity. If it turns out to be an apparent singularity, we shall try to terminate the series solution after a finite number of terms, so that the question of its becoming divergent at $x = 1$ does not arise.

5. Determination of the Nature of the Singularity, at $x = \frac{1}{\beta}$

The diff. eqn. (16) can be put in the form

$$\xi'' + \frac{Q'}{P'}\xi' + \frac{S'}{P'}\xi = 0 \text{ where } P',\ Q',\ S' \text{ are given in the last section.}$$

Now if the eqn. be of the form

$$w'' + \frac{P_1(z)}{(z-a)} w' + \frac{Q_1(z)}{(z-a)^2} w = 0,$$

where $P_1(z)$, $Q_1(z)$ are holomorphic functions at $z = a$, then the first condition (Forsyth, 1902) (3) for the singularity $z = a$ to be apparent is that $P_1(a)$ must be a negative integer greater than zero numerically.

In our case $P_1(x)$ is given by

$$\frac{2\alpha k(1-x^3) + R^3 x^3 (1+2x^3)}{\beta\, x(x^3-1)\,(\beta^2 x^2 + \beta x + 1)}$$

Therefore $P_1\left(\frac{1}{\beta}\right) = \dfrac{2\alpha k\left(1-\frac{1}{\beta^3}\right)^2 + R^3 \frac{1}{\beta^3}\left(1+\frac{2}{\beta^3}\right)}{3\left(\frac{1}{\beta^3}-1\right)}$

Since $\beta^3 = 1 + \dfrac{R^3}{k\alpha}$, we get after some simplification

$$P_1\left(\frac{1}{\beta}\right) = -k\alpha.$$

The first condition is, therefore, satisfied if $(k\alpha)$ is a positive integer greater than zero. We shall see presently that the assumption of this condition leads to the fulfilment of the next condition.

The second condition (Forsyth, 1902) (3) to be satisfied is that the roots of the indicial eqn. are unequal positive integers.

In our case the indicial eqn. is

$$\rho(\rho-1) + \frac{\left[2\alpha k\left(1-\frac{1}{\beta^3}\right)^2 + R^3 \frac{1}{\beta^3}\left(1+\frac{2}{\beta^3}\right)\right]}{3\left(\frac{1}{\beta^3}-1\right)} \cdot \rho = 0$$

or on simplification $\rho(\rho-1) - k\alpha\rho = 0$

or $\rho = 0$ or $(1+k\alpha)$

The roots are unequal positive integers provided $(k\alpha)$ is a positive integer. Hence if the first condition is satisfied, we find that the second condition is also satisfied.

We may assign any convenient value to $(k\alpha)$ only that it should be a positive integer greater than zero. We shall choose it to suit the third and the last condition for the singularity to be apparent.

With the notation in Forsyth (4), we have

$$f_0(\rho) = \rho(\rho-1-k\alpha)$$

so that $\rho_0 = (1+k\alpha),\ \mu = 1,\ \rho_1 = 0;$

We thus have to consider $h_\nu(\rho)$ for $\rho = \rho_1 = 0$; $\nu = \rho_0 - \rho_1 = 1 + k\alpha$.

But

$$g_{(1+k\alpha)}(\rho) = \frac{h_{(1+k\alpha)}(\rho) - g_0(\rho)}{f_0(\rho+1)\, f_0(\rho+2) \ldots f_0(\rho+1+k\alpha)}$$

Now

$$g_\nu(\rho) = \text{coefficient of } x^{\rho+\nu} \text{ in } \sum_{\nu=0}^{\infty} g_\nu x^{\rho+\nu}$$

which is the solution of the diff. eqn.

Evidently therefore for $\rho = \rho_1 = 0$, $g_{(1+k\alpha)}(\rho_1)$ is either a constant or zero.

Let $g_{(1+k\alpha)}(\rho_1) = A\, g_0(\rho_1)$ where A is a constant or zero.

Therefore $h_{(1+k\alpha)}(\rho_1) = A\, f_0(\rho_1+1) \ldots\ldots . f_0(\rho_1+1+k\alpha)$

Now $f_0(\rho+1+k\alpha) = (\rho+1+k\alpha).\,\rho$,

therefore $h_{(1+k\alpha)}(\rho_1) = A\ (\rho_1+1)(\rho_1-k\alpha)(\rho_1+2)(\rho_1+1-k\alpha)\ldots.\rho_1(\rho_1+1+k\alpha)$

Hence $h_{(1+k\alpha)}(\rho_1) = 0$ for $\rho_1 = 0$

This is the *third* condition which is satisfied if $(k\alpha)$ is a positive integer greater than zero.

We, therefore, conclude that the singularity at $x = \frac{1}{\beta}$ is apparent provided $(k\alpha)$ is a positive integer greater than zero

6. Solution of the differential equation (16)

We now proceed to construct a series solution of the eqn. (16) about the origin which is a regular singularity, under the assumption that $(k\alpha)$ is a positive integer.

Substitute in the eqn. (16)

$$\xi = \sum_{n=0}^{\infty} a_n x^{\rho+3n}$$

The indicial equation is given by

$$\rho^2+\rho(2\alpha k-1)-\left(k^2\sigma^2\alpha+\frac{4k^2\alpha^2}{R^3}\right) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

Or

$$\rho = \tfrac{1}{2}\left[-(2k\alpha-1)\pm\left\{(2k\alpha-1)^2+4\left(k^2\sigma^2\alpha+\frac{4k^2\alpha^2}{R^3}\right)\right\}^{\frac{1}{2}}\right]$$

Now $(k\alpha)$ being a positive integer $(1-2k\alpha)$ is negative.

Also the expression under the radical sign is greater than $(2k\alpha-1)$ as is easily seen. Hence the root with negative sign before the radical is negative which we reject, for we want a series in ascending positive powers of x only.

We therefore take

$$\rho = \tfrac{1}{2}\left[-(2k\alpha-1)+\left\{(2k\alpha-1)^2+4\left(k^2\sigma^2\alpha+\frac{4k^2\alpha^2}{R^3}\right)\right\}^{\frac{1}{2}}\right] = \rho_0.$$

Hence $\xi = \sum_{n=0}^{\infty} a_n x^{\rho_0+3n}$ is the solution.

7. Convergence of the series.

We next test the convergence of the series

$$\xi = \sum_{n=0}^{\infty} a_n x^{\rho_0+3n}$$

The recurrence formula is found to be

$$a_n\Big[(\rho+3n)(\rho+3n-1)R^3\beta^3+(\rho+3n)R^3(2k\alpha+2R^3)$$
$$-(2k\alpha R^3+k^2\alpha\sigma^2R^3+2R^6+2R^6k\sigma^2+\frac{\sigma^2}{\alpha}R^9)\Big]$$
$$+a_{n+1}[(\rho+3n+3)(\rho+3n+2)R^3(-1-\beta^3)+(\rho+3n+3)R^3(-4k\alpha+R^3)$$
$$+(4k^2\alpha^2+2R^3k^2\alpha\sigma^2+2k\sigma^2R^6)]$$
$$+a_{n+2}[(\rho+3n+6)(\rho+3n+5)R^3+(\rho+3n+6)R^3\,.\,2k\alpha-(k^2\sigma^2\alpha R^3+4k^2\alpha^2)]$$
$$=0 \quad .. \quad .. \quad (18)$$

Put
$$\frac{a_{n+2}}{a_{n+1}}=N_{n+1}\,;\;\frac{a_{n+1}}{a_n}=N_n\,.$$

Retaining only the highest power of n in (18), we get after dividing by a_{n+1},

$$N_{n+1}=1+\beta^3-\frac{\beta^3}{N_n} \quad .. \quad .. \quad .. \quad .. \quad (19)$$

The difference formula (19) gives

$$N_n=\frac{\left(1-\frac{1}{N_0}\right)(\beta^{3n}+\beta^{3n-3}+\ldots+\beta^3)+1}{\left(1-\frac{1}{N_0}\right)(\beta^{3n-3}+\ldots+\beta^3)+1}$$

Therefore
$$\operatorname*{Lt}_{n\to\infty} N_n=\beta^3$$
$$=1+\frac{R^3}{k\alpha}>1$$

Hence the series $\sum_{n=0}^{\infty} a_nx^{\rho_0+3n}$ is divergent on the surface of the star. We therefore try to terminate the series after a finite number of terms.
Put

$$k\alpha=\theta\,,$$
$$R^3=d\,,$$
$$k\sigma^2=j(j+1)=m \text{ (a positive integer)}.$$

Then
$$\rho^2+\rho(2\theta-1)-\left(m\theta+\frac{4\theta^2}{d}\right)=0 \quad .. \quad .. \quad .. \quad (E)$$

is the indicial equation.
In order that the series may terminate, the coefficients of a_n and a_{n+1} in eqn. (18) must vanish for some finite positive integral value of n.
Coefficient of a_n equated to zero gives

$$(\rho+3n)(\rho+3n-1)\,d\left(1+\frac{d}{\theta}\right)+(\rho+3n)\,d\,(2\theta+2d)$$
$$-\left(2\theta d+md.\theta+2d^2+2md^2+\frac{m}{\theta}d^3\right)=0 \quad .. \quad .. \quad .. \quad (F)$$

Coefficient of a_{n+1} equated to zero gives

$$(\rho+3n+3)\,(\rho+3n+2)\,d\left(-2-\frac{d}{\theta}\right)+(\rho+3n+3)\,d\,(-4\theta+d)+(4\theta^2+2md\theta+2md^2) = 0 \quad .. \quad .. \quad \text{(H)}$$

Our object is to evaluate θ with the help of (E), (F) and (H) and to see if these equations are compatible with our assumption that $\theta = k\alpha$ is a positive integer greater than zero, so that $x = \frac{1}{\beta}$ becomes an apparent singularity.

Simplification with the help of (E) reduces (F) and (H) to the forms

$$md^2-d\,\{6n\rho+3n(3n-1)+(6n-2)\theta\}\,-4\theta^2 = 0 \quad .. \quad .. \quad (\mathrm{F}_1)$$

$$d^2\left[\left(-\frac{6n}{\theta}-\frac{6}{\theta}+3\right)\rho-\frac{9n^2}{\theta}-\frac{15n}{\theta}+3n-\frac{6}{\theta}+3-m\right]$$

$$+d[-12\rho-36n-12-20\theta]+4\theta^2 = 0 \quad .. \quad .. \quad .. \quad .. \quad (\mathrm{H}_1)$$

Equating the coefficient of d^2 and d in these equations, we get

$$\frac{1}{m}\left[\left(-\frac{6n}{\theta}-\frac{6}{\theta}+3\right)\rho-\frac{9n^2}{\theta}-\frac{15n}{\theta}+3n-\frac{6}{\theta}+3-m\right]$$

$$= \frac{(-12\rho-36n-12-20\theta)}{[-6n\rho-3n(3n-1)-(6n-2)\theta]}$$

$$= -1$$

From these equations we obtain

$$\left(-\frac{6n}{\theta}-\frac{6}{\theta}+3\right)\rho-\frac{9n^2}{\theta}-\frac{15n}{\theta}+3n-\frac{6}{\theta}+3 = 0$$

and

$$(-12\rho-36n-12-20\theta) = 6n\rho+3n(3n-1)+(6n-2)\theta$$

These reduce to

$$(\theta-2n-2)\rho-(3n^2+5n+2)+3(n+1)\theta = 0$$
$$(6n+12)\rho+3n(3n+11)+12+(6n+18)\theta = 0$$

Elimination of ρ between these two equations gives the following equation in θ alone

$$\theta^2(2n+6)+\theta(5n^2+13n+4)+(2n+4)(3n^2+5n+2) = 0$$

Since all the coefficients are positive numbers, there is no positive root of the equation. Hence if (F) and (H) held then θ cannot be a positive integer.

Therefore, if $x = \frac{1}{\beta}$ is an apparent singularity, so that $\theta = k\alpha =$ a positive integer greater than zero, (F) and (H) cannot hold with the consequence that the series cannot be made to terminate after a finite number of terms.

Hence the series diverges on the surface of the star and gives an infinite amplitude of oscillation there.

We therefore, conclude that under the conditions imposed, the non-radial oscillations are unstable and the star in question will throw out expanding gaseous material in case some such oscillations set in.

8. Conclusion

Spectroscopic observations have revealed several expanding nebulae. The Crab Nebula is still expanding at the rate of about 800 miles per second. If this expansion is due to the explosion of a super-nova, then that phenomena must have occurred about 900 years ago. Indeed a new star in that position was recorded in 1054 A.D. by Chinese observers. The Filamentary Nebula in the constellation Cygnus shows an expansion with an angular velocity of 0·05 second per year, so that the expansion must have begun about 100,000 yrs. ago. But there was no astronomer then to record the appearance of a new star. The expanding nature of these nebulae is surmised by some astronomers to be the result of nova-explosion in the remote past. But nothing can be said with complete certainty. It has also been suggested that the planetary nebulae are the results of Novae-explosion, but there are good reasons for doubting this theory. The number of Planetary Nebulae is comparatively small, whereas the Nova-explosions are much too frequent.

The investigation carried out in the present paper suggests that if somehow non-radial oscillations are set in in centrally condensed stars consisting of gas, the ratio of whose specific heats is 1·5, the oscillations tend to become unstable.* Consequently the star will throw out columns or shells of expanding gas with considerable velocity.

On the strength of the result of the present investigation and that of Pekeris it is suggested that the expanding nebulae may have their origin in the unstable non-radial oscillations in stars which are either centrally condensed or homogeneous. These oscillations may have originated due to some internal disturbance in the star or due to the disturbance caused by the nearby passage of another star.

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Summary

The differential equations governing small non-radial adiabatic oscillations of gaseous spheres are derived correctly to the order of accuracy to which changes in the gravitational potential produced by such oscillations can be ignored. They are then applied to the Roche Model in which practically the whole mass is concentrated at the centre and in the surrounding atmosphere density varies as the inverse square of the distance from the centre. It has been shown that for the general value of γ—the ratio of specific heats, the problem seems to be intractable. Next the case for the particular value of $\gamma = 1{\cdot}5$ is considered and it is shown that with a certain restriction on the form of the frequency function the amplitude of oscillation diverges from zero at the centre of the star to infinity at the surface. The oscillations are thus unstable.

On the basis of this investigation, it has been suggested that the expansion of the Crab Nebula, the Filamentary Nebulae, the Planetary Nebulae and others may have its origin in the instability of non-radial oscillations that might have started in the remote past in centrally condensed stars due to some internal cause or due to the nearly passage of another star.

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* Pekeris (1938) has established the instability of Non-radial oscillations in a homogeneous star, for all modes.

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