A NOTE ON THE KARMAN'S SPECTRUM FUNCTION OF ISOTROPIC TURBULENCE

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Introduction

If $F(k, t)$ be the spectrum function of isotropic turbulence, the decay of turbulence, according to the theory proposed by Heisenberg (1949), takes place according to the equation

$$-\frac{\partial}{\partial t}\int_0^k F(k', t)\, dk' = 2\,(\nu+\eta_k)\int_0^k F(k', t)\, k'^2\, dk', \quad \ldots \quad \ldots \quad (1)$$

where ν is the kinematic viscosity, and η_k turbulent viscosity, for which Heisenberg (1949) assumed the form

$$\eta_k = K\int_k^\infty \sqrt{\frac{F(k', t)}{k'^3}}\, dk' \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

A self-preserving solution of this equation in the form

$$F(k, t) \sim \frac{1}{\sqrt{t}}\, f(k\sqrt{t}) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

was given by Heisenberg, which has the property that $f(x) \sim x$, for small x and $f(x) \sim x^{-\frac{5}{3}}$ for large x. They correspond to the linear and Kolmogoroff parts of the spectrum respectively. The above solution was investigated more completely by Chandrasekhar (1949) who has given numerical tables for $f(x)$. While solution (3) gives a complete similarity throughout the spectrum, it gives neither Loitsiansky's result (1939), nor the ultimate law of dissipation with time as $(t-t_0)^{-\frac{5}{2}}$, observationally verified by Batchelor and Townsend (1948).

N. R. Sen (1951) has recently shown that at the initial stages when the low frequency eddies are predominant (ν small compared to η_k) equations (1) and (2) admit a more general type of solution of the form

$$F(k, t) \sim \frac{1}{K^2}\cdot\frac{1}{k_0^3 t_0^2}\left(\frac{t_0}{t}\right)^{2-3c} f\left((k/k_0)(t/t_0)^c\right) \ldots \quad \ldots \quad \ldots \quad (4)$$

where c is an arbitrary constant. Limiting c to $< \frac{2}{3}$, it can be used as a parameter determining a family of early period solutions. It was shown by Sen that this solution has the following characteristics:

$$f(x) \approx \text{const.}\ x^{(2-3c)/c}, \text{ as } x \to 0\,;$$

and

$$f(x) \approx \text{const.}\ x^{-5/3}, \text{ as } x \to \infty$$

whatever c may be.

While $c = \frac{1}{2}$ gives Heisenberg's results, the above solution (4) is remarkable in the respect that for $c = \frac{2}{7}$, it gives on the one hand Loitsiansky's result, viz.,

(i) $F \sim$ const. k^4 as $k \to 0$, and on the other hand the decay-law and characteristic length * namely $u^2 \sim$ const. $t^{-10/7}$, and $\lambda^2 \sim 7\nu t$ respectively, which were first pointed out by Kolmogoroff (1941) to be true when Loitsiansky's result is assumed. Sen's solution (4) shows that all the above three conclusions follow from this self-preserving solution (4) of Heisenberg's equations (1), (2), without any ad hoc assumption of Loitsiansky's result. The initial lack of isotropy at $k \to 0$ is, of course, outside the scope of the present solution in which similarity has been assumed. It has been suggested that the parameter c may be connected with the mode of excitation of turbulence.

The object of this note is to point out that the *form* of the spectrum (4) given by Sen is also associated with a more general decay equation suggested by Karman in 1948.

2. Self-preserving Solution of Decay Equation with Karman's Spectrum Function

Assuming the existence of a transition function for energy between the intervals dk and dk' (which depends on the energy density and wave numbers k and k'), Karman (1948) obtained by dimensional reasoning the following equation for the decay of the spectrum function

$$-\frac{\partial F}{\partial t} = C\left[\{F(k)\}^{\alpha} k^{\beta} \int_0^k \{F(k')\}^{\frac{3}{2}-\alpha} k'^{\frac{1}{2}-\beta}\, dk' - \{F(k)\}^{\frac{3}{2}-\alpha} k^{\frac{1}{2}-\beta} \int_k^{\infty} \{F(k')\}^{\alpha} k'^{\beta}\, dk'\right] - 2\nu k^2 F \quad \ldots \quad \ldots \quad (5)$$

where ν is the molecular viscosity, α, β, as yet unspecified constants, and C an absolute constant. When the first term on the right of (5) is entirely negligible the form would include Heisenberg's equation (1), (2) for $\alpha = \frac{1}{2}$, $\beta = -\frac{3}{2}$.

For steady state under negligible molecular viscosity, (5) has been shown to give the Kolmogoroff spectrum by Karman.

When the turbulent viscosity is dominant the above equation will have the form

$$-\frac{\partial F}{\partial t} = C\left[F^{\alpha} k^{\beta} \int_0^k F^{\frac{3}{2}-\alpha} k'^{\frac{1}{2}-\beta}\, dk' - F^{\frac{3}{2}-\alpha} k^{\frac{1}{2}-\beta} \int_k^{\infty} F^{\alpha} k'^{\beta}\, dk'\right] \quad \ldots \quad (5a)$$

Let us seek a self-preserving solution of this equation in the form

$$F = \frac{1}{C^2} \cdot \frac{\{u(\lambda)\}^q \cdot \{s(\tau)\}^p}{k_0^3 t_0^2 \cdot \tau^r} \cdot f(\lambda \cdot s(\tau)) \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

where p, q, r are constants, $\lambda = k/k_0$, $\tau = t/t_0$, $u(\lambda)$ a function of λ and $s(\tau)$, a function of τ. On substitution of (6) in (5a) and simplification one obtains the relation

$$\int_0^x u^q(\lambda)\left[r - \frac{p \cdot \tau \cdot s_\tau}{s}\right] f(x)\, dx - \int_0^x u^q(\lambda) \cdot \tau \cdot s_\tau \cdot \frac{x}{s} \cdot \frac{df}{dx}\, dx$$

$$= \int_0^{\infty} u^{q\alpha}(\lambda) \cdot x^{\beta} \cdot f^{\alpha}(x)\, dx \times \int_0^x u^{(\frac{3}{2}-\alpha)q} \cdot \frac{s^{\frac{p-3}{2}}}{\tau^{\frac{r-2}{2}}} \cdot x^{\frac{1}{2}-\beta} f^{\frac{3}{2}-\alpha}(x)\, dx \quad \ldots \quad (7)$$

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* This last result was pointed out to me by Prof. Sen.

where $x = s\,.\,\lambda$, and s_τ denotes $\frac{ds}{d\tau}$. The condition of similarity will be satisfied if

$$\left.\begin{array}{l} q = 0 \\ s = \alpha'\tau^{c} \\ r = (p-3)c+2 \end{array}\right\} \qquad (8)$$

where α' is the constant of integration of

$$\tau \cdot \frac{s_\tau}{s} = c,$$

c being another constant. The interesting point here is that equations (8) do not involve α and β. The argument of the function f in (6) does not depend on α and β, but its *form* does. On substitution of (8), (7) will have the form

$$(2-3c)\int_0^x f(x)\,dx - c\int_0^x x\frac{df(x)}{dx}\,dx$$

$$= \int_x^\infty x^\beta f^\alpha(x)dx \times \int_0^x x^{\frac{1}{2}-\beta} f^{\frac{3}{2}-\alpha}(x)\,dx \qquad (9)$$

Again substituting the values of q, s, r from (8) in (6), one finds that the *form* of the self-preserving solution $F(k, t)$ of Karman's equation (5a) is the same as (4) given by Sen. The constituent function $f(x)$ of self-preservation is now given by a different equation, namely (9), which involves α and β.

It may be easily shown that equation (9) admits of similar solutions of the form $h^3f(hx)$, $f(x)$ being any solution of the same equation, and h any constant (which is unconnected with α and β). This similarity property is known also to belong to the solution of Heisenberg's equation.

When the inertia terms in the equation of decay (1) are comparable to the viscous term, i.e., the viscosity part can no longer be neglected, the equation of decay of turbulent energy will have the general form (5). If we assume a solution of the form (8) for the complete equation (5), the similarity condition will be satisfied for $s = \alpha'\,\tau^c$, only if $c = \frac{1}{2}$. This again means that Heisenberg's form of solution (3) is the only similar solution which is valid for the entire general Karman spectrum of turbulence.

We shall now examine the behaviour of $f(x)$ satisfying equation (9) as $x \to 0$, and also as $x \to \infty$.

(a) Asymptotic behaviour of $f(x)$ at $x \to 0$.

If

$$f(x) \sim Ax^n \quad (x \to 0) \qquad (10)$$

then proceeding in the usual way we find the following equation for n,

$$n^2\left(\frac{c}{2}-\alpha c\right)+n\left\{(2\alpha-1)+c(2-\beta-3\alpha)\right\}+(2-3c)(\beta-\tfrac{1}{2}) = 0 \qquad (11)$$

This quadratic equation in n has two solutions, viz.,

$$n_1 = \frac{2-3c}{c} \qquad (12.1)$$

and,

$$n_2 = \frac{2\beta-1}{1-2\alpha} \qquad (12.2)$$

N. R. Sen (1951) obtained the asymptotic solution (12.1) (which is free from α, β) for the Heisenberg spectrum. It is to be noted that in the case $\alpha = \frac{1}{2}$, $\beta = -\frac{3}{2}$ (Heisenberg spectrum) the above equation (11) becomes linear, which gives the root

$$n_1 = \frac{2-3c}{c}$$

agreeing with Sen's result; for $c = \frac{2}{7}$, n_1 is equal to 4. We take this value of c in our subsequent argument. In any other case different from that of Heisenberg, (12.1) is a root, which we take as equal to 4 ($c = \frac{2}{7}$). There is a second root $n_2 = \frac{2\beta-1}{1-2\alpha}$. In order that n_2 may be positive, we should have any one of the two alternatives :—

$$\text{(i) } \alpha > \tfrac{1}{2}\,;\ \beta < \tfrac{1}{2} \text{ or (ii) } \alpha < \tfrac{1}{2}\,;\ \beta > \tfrac{1}{2}.$$

(i) when $\alpha > \frac{1}{2}$; $\beta < \frac{1}{2}$, the expansion of $f(x)$, for $x \to 0$ will begin with x^4, if $4\alpha+\beta < \frac{5}{2}$; but if $4\alpha+\beta > \frac{5}{2}$, the expansion of $f(x)$, as $x \to 0$ begins with x^{n_2}, n_2 being < 4. These are really the conditions given by Karman.

(ii) But when $\alpha < \frac{1}{2}$; $\beta > \frac{1}{2}$, the above statements are reversed. Karman's conditions have validity for the first alternative only.

(b) Asymptotic behaviour of $f(x)$ for $x \to \infty$.

Putting $f(x') = e^{-\omega(x')}$ and following Heisenberg's method of approximation the behaviour of $f(x)$, for $x \to \infty$, will be given by the behaviour at large x of f, when it is governed by the following equation :—

$$\frac{d}{dx}\left\{(2-3c)\,x^{-\beta}f^{1-\alpha} - cx^{1-\beta}f^{-\alpha}f' + \frac{x^{\frac{3}{2}-\beta}f^{\frac{3}{2}-\alpha}}{(\beta+1)+\alpha x\frac{f'}{f}}\right\} + x^{\frac{1}{2}-\beta}f^{\frac{3}{2}-\alpha} = 0 \quad .. \quad (13)$$

provided we assume that $\qquad \alpha\omega'-\beta-1 > 0 \qquad .. \quad .. \quad .. \quad .. \quad (14)$

(ω' meaning differentiation with respect to $\log x$). If we seek a solution such as

$$f(x) \sim Ax^{-n}\,(x \to \infty),\ A \neq 0 \quad .. \quad .. \quad .. \quad (15)$$

where n is positive, (13) gives the following equation (where only the highest order terms are retained)

$$x^{-[\beta-\frac{1}{2}+n(\frac{3}{2}-\alpha)]}A^{\frac{3}{2}-\alpha}\left[\frac{\frac{1}{2}-\beta}{(\beta+1)-\alpha n} - (\tfrac{3}{2}-\alpha)\frac{n}{(\beta+1)-\alpha n} + 1\right]$$
$$+x^{-[n(1-\alpha)+\beta+1]}A^{1-\alpha}\left[n^2(\alpha c-c)+n\left\{(3c-2)\,(1-\alpha)-\beta c\right\}+(3c-2)\beta\right]$$
$$+\ \ldots\ldots = 0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (16)$$

If the first term be the significant one (i.e. algebraically be with higher power of x)

$$\beta-\tfrac{1}{2}+n(\tfrac{3}{2}-\alpha) < n(1-\alpha)+\beta+1$$

or

$$n < 3\,;$$

then the solution will be given by equating the coefficient of

$$x^{-[\beta-\frac{1}{2}+n(\frac{3}{2}-\alpha)]}$$

to zero, which gives $n = \frac{5}{3}$.

If we assume the second term to be dominant, then $n > 3$, and vanishing of th coefficient in the second term would give two values

$$n = \beta/(\alpha-1), \text{ or } (3c-2)/c.$$

The second of these two values is now a violation of the condition $n > 3$, as c is restricted to have a value less than $\frac{2}{3}$.

We note that for the assumption (15), the inequality (14) reduces to

$$n\alpha - \beta - 1 > 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \ (14a)$$

This is not violated in any of the above arguments, but is a restriction on α and β.

Under this restriction we have asymptotic expansion of $f(x)$ for $x \to \infty$ as, either

$$(1)\ f(x) \approx Ax^{-\frac{5}{3}}$$

or

$$(2)\ f(x) \approx Ax^{-\beta/(\alpha-1)} \quad (\beta/(\alpha-1) > 3)\,.$$

The first is the well-known Kolmogoroff spectrum but the second is also a mathematical possibility for equation (5a). If we, however, closely analyze (5a), and (16), we note that the first term in the latter equation is contributed by the two terms on the right hand side of equation (5a), and the second term of (16) by the term on the left side of (5a). Hence in case (1) above when the solution approaches the Kolmogoroff spectrum, each of the two terms on the right of (5a) makes finite contribution but their difference is a small term of an order which is neglected. The steady condition in the spectrum is reached by any spectral band receiving from outside and also transmitting to the outside finite and equal amounts of energies. The entire spectrum is far from the decaying stage. But when condition (2) is satisfied each of these two amounts of energies is at least a small quantity of the first order. The spectrum then is very near its state of complete decay.

Summary

The general self-preserving *form* of the spectrum function governed by Heisenberg's equation for decay of turbulence at the stage when the effect of molecular viscosity is negligible, recently given by N. R. Sen, is found to be valid under the same condition for the more general decay equation proposed by Karman. The *form* of the constituent function f of self-preservation depends on the arbitrary constants α and β introduced by Karman. The asymptotic behaviour of the self preserving solution of Heisenberg's equation as $k \to 0$ still remains valid under certain inequality condition satisfied by α and β in the general case of Karman; but under the opposite condition a second behaviour of the asymptotic solution at $k \to 0$ is possible. For $k \to \infty$ the Kolmogoroff spectrum is obtained as a (transient) steady state condition with finite transfer of equal amounts of energies between a spectrum band and its outside, but for Karman's spectrum function when $\beta/(\alpha-1)>3$, a second steady state behaviour becomes mathematically possible at near the complete decay end of the spectrum.

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