APPELL SET OF POLYNOMIALS

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1. INTRODUCTION

A set of polynomials $\{P_n(x)\}$, where $P_n(x)$ is of degree n in x, is said to be an Appell set, if it satisfies the relation

$$P'_n(x) = P_{n-1}(x). \qquad (1)$$

An equivalent definition is the existence of a power series

$$A(t) = \sum_{n=0}^{\infty} a_n t^n \quad a_0 \neq 0 \qquad (2)$$

such that

$$A(t)\, e^{tx} = \sum_{n=0}^{\infty} t^n P_n(x) \qquad (3)$$

and $A(t)$ is called the generating function of the set of polynomials $P_n(x)$.

Varma [4] has recently given a generalization of Sheffers [1] representation of Appell polynomials. His result may be stated as follows:

If

$$I_{n,\,r} = \int_0^{\infty} \delta_n(t)\, t^r\, d\beta(t) \qquad (4)$$

$$n,\, r = 0,\, 1,\, 2, \ldots.$$

exists such that $I_{0,\,0} \neq 0$, then the Appell set is given by

$$P_n(x) = \int_0^{\infty} K_n(x, t)\, d\beta(t) \qquad (5)$$

where

$$K_n(x, t) = \sum_{r=0}^{n} \delta_{n-r}(t) \frac{(x+t)^r}{r!}. \qquad (6)$$

In particular

$$P_n(x) = \frac{x^n}{n!} \int_0^{\infty} {}_3F_2 \left\{ \begin{matrix} -n, a, b \\ c, d \end{matrix} ; -\frac{t}{x} \right\} d\beta(t)$$

and the generating function then is

$$A(t) = \int_0^{\infty} {}_2F_2 \left\{ \begin{matrix} a, b \\ c, d \end{matrix} ; ut \right\} d\beta(u)$$

In the present paper we give an extension of Varma's result to

$$P_n(x) = \frac{x^n}{n!}\int_0^\infty {}_{r+1}F_r\left\{\begin{matrix}-n, a_1, a_2, \dots a_r\\ b_1, b_2, \dots b_r\end{matrix}; -\frac{t}{x}\right\} d\beta(t)$$

and

$$A(t) = \int_0^\infty {}_rF_r\left\{\begin{matrix}a_1, a_2, \dots a_r\\ b_1, b_2, \dots b_r\end{matrix}; ut\right\} d\beta(u)$$

and also give, by means of operational calculus, some expressions for $P_n(x)$ in terms of confluent hypergeometric functions.

2. A New Integeral Representation

We begin by establishing the following lemma:

$$\sum_{n=0}^\infty \frac{(xu)^n}{n!}\, {}_{r+1}F_r\left\{\begin{matrix}-n, a_1, a_2, \dots a_r\\ b_1, b_2, \dots b_r\end{matrix}; -\frac{t}{x}\right\}$$
$$= e^{xu}\, {}_rF_r\left\{\begin{matrix}a_1, a_2, \dots a_r\\ b_1, b_2, \dots b_r\end{matrix}; ut\right\}. \quad \dots \quad \dots \quad \dots \quad (7)$$

Proof: with the usual notation

$$(a_p, r) = a_p(a_p+1) \dots (a_p+r-1)$$

we can write the left hand side of (7) as

$$\sum_{n=0}^\infty \frac{(xu)^n}{n!} \sum_{p=0}^n (-n, p) \frac{(a_1, p) \dots (a_r, p)}{(b_1, p) \dots (b_r, p)} \frac{1}{p!}\left(-\frac{t}{x}\right)^p.$$

Now

$$(-n, p) = (-)^n n(n-1) \dots (n-p+1)$$
$$= (-)^n n!/(n-p)!,$$

therefore the left hand side

$$= \sum_{n=0}^\infty \frac{(xu)^n}{n!} \sum_{p=0}^n \frac{(a_1, p) \dots (a_r, p)}{(b_1, p) \dots (b_r, p)} \frac{n!}{p!\,(n-p)!}\left(\frac{t}{x}\right)^p$$

$$= \sum_{p=0}^\infty \frac{(ut)^p}{p!} \frac{(a_1, p) \dots (a_r, p)}{(b_1, p) \dots (b_r, p)} \sum_{n=p}^\infty \frac{(xu)^{n-p}}{(n-p)!}$$

$$= e^{xu}\, {}_rF_r\left\{\begin{matrix}a_1, a_2, a_3, \dots a_r\\ b_1, b_2, b_3, \dots b_r\end{matrix}; ut\right\},$$

by an inversion of the order of summation.

Rice's result [3]

$$\sum_{n=0}^\infty \frac{(xu)^n}{n!}\, {}_3F_2\left\{\begin{matrix}-n, a_1, a_2\\ b_1, b_2\end{matrix}; -\frac{t}{x}\right\} = e^{xu}\, {}_2F_2\left\{\begin{matrix}a_1, a_2,\\ b_1, b_2,\end{matrix}; ut\right\}$$

follows at once from our lemma by taking $r = 2$.

Our lemma now gives that

$$\frac{x^n}{n!}\,{}_{r+1}F_r\left\{\begin{matrix}-n, a_1, a_2, \ldots. a_r\\ b_1, b_2, \ldots. b_r\end{matrix}; -\frac{t}{x}\right\}$$

is an Appell polynomial.

If we write the equivalent series for ${}_{r+1}F_r\left\{\begin{matrix}-n, a_1, a_2, \ldots. a_r\\ b_1, b_2, \ldots. b_r\end{matrix}; -\frac{t}{x}\right\}$ we get after some rearrangement

$$\begin{aligned}\frac{x^n}{n!}\,{}_{r+1}F_r&\left\{\begin{matrix}-n, a_1, a_2, \ldots. a_r\\ b_1, b_2, \ldots. b_r\end{matrix}; -\frac{t}{x}\right\}\\
&=\sum_{p=0}^{n}\frac{(a_1,p)\ldots.(a_r,p)}{(b_1,p)\ldots.(b_r,p)}\frac{x^{n-p}\,t^p}{p!\,(n-p)!}\\
&=\sum_{p=0}^{n}\frac{(a_1,p)\ldots.(a_r,p)}{(b_1,p)\ldots.(b_r,p)}\sum_{q=0}^{n-p}\frac{(x+t)^{n-p-q}}{(n-p-q)!}\frac{(-t)^{p+q}(-)^p}{q!\,p!}\\
&=\sum_{q=0}^{n}\frac{(x+t)^{n-q}}{(n-q)!}(-t)^q\sum_{p=0}^{q}(-)^p\frac{(a_1,p)\ldots.(a_r,p)}{(b_1,p)\ldots.(b_r,p)}\frac{1}{p!\,(q-p)!}\\
&=\sum_{q=0}^{n}\delta q\,(t)\,\frac{(x+t)^{n-q}}{(n-q)!}, \qquad (8)\end{aligned}$$

where

$$\delta q(t)=\frac{(-t)^q}{q!}\sum_{p=0}^{q}(-)^p\frac{(a_1,p)\ldots.(a_r,p)}{(b_1,p)\ldots.(b_r,p)}\binom{q}{p}(-)^p. \qquad (9)$$

Hence from (6)

$$\begin{aligned}P_n(x)&=\frac{x^n}{n!}\int_0^\infty {}_{r+1}F_r\left\{\begin{matrix}-n, a_1, a_2, \ldots. a_r\\ b_1, b_2, \ldots. b_r\end{matrix}; -\frac{t}{x}\right\}d\beta(t)\\
&=\sum_{p=0}^{n}\frac{(a_1,p)\ldots.(a_r,p)}{(b_1,p)\ldots.(b_r,p)}\frac{x^{n-p}}{(n-p)!}\frac{\mu_p}{p!} \qquad (10)\end{aligned}$$

is an Appell polynomial. Here $\{\mu_p\}$ is the set of moment constants and $\beta(t)$ is the type of function considered by Sheffer.

On account of the identity

$${}_{r+1}F_r\left\{\begin{matrix}-n, a_1, a_2, \ldots. a_r\\ a_1, a_2, \ldots. a_r\end{matrix}; -u\right\}=(1+u)^n,$$

the function defined by (10) reduces to Appell polynomials given by Sheffer when we take $a_p = b_p$ $(p = 1, 2, 3, \ldots., r.)$.

3. Generating function

The generating function of the polynomials given by (10) will be

$$\sum_{n=0}^{\infty}u^n P_n(x)=\sum_{n=0}^{\infty}\frac{(xu)^n}{n!}\int_0^\infty {}_{r+1}F_r\left\{\begin{matrix}-n, a_1, a_2, \ldots. a_r\\ b_1, b_2, \ldots. b_r\end{matrix}; -\frac{t}{x}\right\}d\beta(t).$$

On interchanging the order of integration and summation and using our lemma we get

$$A(u)\, e^{ux} = \sum_{n=0}^{\infty} P_n(x)\, u^n$$

$$= e^{xu} \int_0^{\infty} {}_rF_r \left\{ \begin{matrix} a_1, a_2, \ldots . a_r \\ b_1, b_2, \ldots . b_r \end{matrix} ; ut \right\} d\beta(t) .$$

Therefore

$$A(u) \doteq \int_0^{\infty} {}_rF_r \left\{ \begin{matrix} a_1, a_2, \ldots . a_r \\ b_1, b_2, \ldots . b_r \end{matrix} ; ut \right\} d\beta(t) . \qquad \ldots \quad \ldots \quad (11)$$

It is easy to see that

$${}_rF_r \left\{ \begin{matrix} a_1, a_2, \ldots . a_r \\ b_1, b_2, \ldots . b_r \end{matrix} ; ut \right\}$$

can be written as

$${}_rF_r \left\{ \begin{matrix} a_1, a_2, \ldots . a_r \\ b_1, b_2, \ldots . b_r \end{matrix} ; ut \right\} = \prod_{p=1}^{r} \frac{\Gamma(b_p)}{\Gamma(a_p)\Gamma(b_p-a_p)} \int_0^1 x_p^{a_p-1}(1-x_p)^{b_p-a_p-1} e^{ut-\prod_1^r x_p} dx_p.$$

$$\ldots \quad (12)$$

$$R(a_p) > 0, \quad R(b_p-a_p) > 0 .$$

If we denote the generating function in (11) by $A_r(u)$ and use (12) we get

$$A_r(u) = \prod_{p=1}^{r} \frac{1}{B(a_p, b_p-a_p)} \int_0^1 x_p^{a_p-1} (1-x_p)^{b_p-a_p-1} dx_p \int_0^{\infty} e^{ut-\prod_1^r x_p} d\beta(t)$$

$$= \prod_{p=r-q+1}^{r} \frac{1}{B(a_p, b_p-a_p)} \int_0^1 x_p^{a_p-1} (1-x_p)^{b_p-a_p-1} A_{r-q}\left(u \prod_{p=r-q+1}^{r} x_p\right) dx_p, \quad (13)$$

which gives us a relation between the generating functions obtained by the different choice of the order of the hypergeometric functions.

4. Relation between Appell polynomials and some known functions

If x and y be two independent variables it is easy to see by Taylor's theorem and property of Appell polynomials that

$$P_n(x+y) = \sum_{p=0}^{n} \frac{y^p}{p!} P_{n-p}(x) , \qquad \ldots \quad \ldots \quad \ldots \quad (14)$$

and in particular

$$P_n(x) = \sum_{r=0}^{n} \frac{x^r}{r!} P_{n-r}(0) \ldots \qquad \ldots \quad \ldots \quad \ldots \quad (15)$$

If we multiply both sides of (15) by e^{-px} and integrate from 0 to ∞, we get

$$p \int_0^{\infty} e^{-px} P_n(x)\, dx = \sum_{r=0}^{n} P_{n-r}(0) \frac{1}{p^r} . \ldots \qquad \ldots \quad \ldots \quad (16)$$

In the notation of operational calculus we can write (16) as

$$P_n(x) \doteqdot \sum_{r=0}^{n} P_{n-r}(0)\, p^{-r}. \qquad \ldots \quad \ldots \quad \ldots \quad (17)$$

4.1. By means of an important theorem in operational calculus we have from (17)

$$e^{-x} P_n(x) \doteqdot \sum_{r=0}^{x} P_{n-r}(0)\, p/(p+1)^{r+1} \ldots \quad \ldots \quad \ldots \quad (18)$$

But we know that [Humbert and McLachlan 6]

$$x^{-\nu-\frac{1}{2}} e^{-x} W_{\mu,\nu}(x) \doteqdot (-)^{\mu+\nu+\frac{1}{2}} \Gamma(\mu-\nu+\tfrac{1}{2}) \frac{p^{\mu+\nu-\frac{1}{2}}}{(1+p)^{\mu-\nu+\frac{1}{2}}}.$$

Hence taking

$$\mu = \frac{r+2}{2} \text{ and } \nu = -\frac{r-1}{2},$$

we get

$$x^{\frac{r-2}{2}} e^{-x/2} W_{\frac{r+2}{2},\, -\frac{r-1}{2}}(x) \doteqdot \Gamma(r+1) p/(p+1)^{r+1}.$$

Now from (18) and using Lerch's theorem we get

$$e^{-x} P_n(x) = \sum_{r=0}^{n} \frac{P_{n-r}(0)}{r!} x^{\frac{r-2}{2}} e^{-x/2} W_{\frac{r+2}{2},\, -\frac{r-1}{2}}(x),$$

and finally

$$P_n(x) = \sum_{r=0}^{n} \frac{P_{n-r}(0)}{r!} x^{\frac{r-2}{2}} e^{x/2} W_{\frac{r+2}{2},\, -\frac{r-1}{2}}(x). \qquad \ldots \quad \ldots \quad (19)$$

4.2. From (17) as before we have

$$e^{-x/2} P_n(x) \doteqdot \sum_{r=0}^{n} P_{n-r}(0)\, p/(p+\tfrac{1}{2})^{r+1} \ldots \quad \ldots \quad \ldots \quad (20)$$

But we know that [6]

$$M_{\mu,\nu}(t) \doteqdot \Gamma(\nu+3/2) \frac{p}{(p+\frac{1}{2})^{r+3/2}} {}_2F_1\left\{\nu+3/2,\, -\mu+\nu+\tfrac{1}{2};\, \frac{1}{p+\frac{1}{2}}\right\},$$

$$R(\nu) > -3/2.$$

Putting $\nu = r-\frac{1}{2}$ and $\mu = r$ we get

$$M_{r,\, r-\frac{1}{2}}(t) \doteqdot \Gamma(r+1)\, p/(p+\tfrac{1}{2})^{r+1},$$

and hence as before

$$P_n(x) = \sum_{r=0}^{n} \frac{P_{n-r}(0)}{r!} M_{r,\, r-\frac{1}{2}}(x)\, e^{x/2}. \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

4.3. It is well known from operational calculus that if $f(t) \doteqdot \phi(p)$ then

$$\frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-x^2/4t} f(x) dx \doteqdot \phi(\sqrt{p}).$$

Therefore from (17) we have

$$\frac{1}{\sqrt{\pi x}}\int_0^\infty \exp\left[-s-\frac{s^2}{4x}\right]P_n(s)ds \doteqdot \sum_{r=0}^{n} P_{n-r}(0)\frac{\sqrt{p}}{(1+\sqrt{p})^{r+1}}. \quad .. \quad (22)$$

But we know that [6]

$$\sqrt{\frac{2}{\pi}}(2x)^{\frac{1}{2}(n-1)}e^{x/2}D_{-n}(\sqrt{2}\,x) \doteqdot \frac{\sqrt{p}}{(1+\sqrt{p})^n},$$

where $D_n(x)$ is Weber's parabolic cylindrical function. Therefore as before

$$\frac{1}{\sqrt{\pi x}}\int_0^\infty \exp\left[-s-\frac{s^2}{4x}\right]P_n(s)\,ds = \sqrt{\frac{2}{\pi}}\sum P_{n-r}(0)(2x)^{r/2}e^{x/2}D_{-(r+1)}(\sqrt{2}\,x),$$

and finally

$$\int_0^\infty e^{-s-s^2/4x}P_n(s)ds = \sum P_{n-r}(0)\,(2x)^{\frac{r+1}{2}}\,e^{x/2}D_{-(r+1)}(\sqrt{2}\,x). \quad .. \quad (23)$$

4.4. The set of polynomials $\{\Pi_n(x)\}$, where $\Pi_n(x)$ is a polynomial of degree n in x defined by the relation

$$\Pi_n(x) = e^x\left(\frac{d}{dx}\right)^n\{e^{-x}A_n(x)\}, \quad .. \quad .. \quad .. \quad (24)$$

where

$$A_n(x) = (a_0, a_1, \ldots .a_n, x)^n,$$

is called Angelescus' set.

It is quite apparent that $\frac{A_n(x)}{n!}$ behaves like an Appell polynomial, and so from (24) we get

$$\frac{\Pi_n(x)}{n!} = e^x\left(\frac{d}{dx}\right)^n\left\{e^{-x}P_n(x)\right\}. \quad .. \quad .. \quad .. \quad (25)$$

Taking the integral representation for $P_n(x)$ from (10) we get

$$\frac{\Pi_n(x)}{n!} = e^x\sum_{p=0}^{n}\frac{1}{p!}\frac{(a_1,p)\ldots .(a_r,p)}{(b_1,p)\ldots .(b_r,p)}\left(\frac{d}{dx}\right)^n\left(\frac{e^{-x}x^{n-p}}{(n-p)!}\right)\mu_p$$

$$= \sum_{p=0}^{n}\frac{1}{p!}\mu_p\frac{(a_1,p)\ldots .(a_r,p)}{(b_1,p)\ldots .(b_r,p)}L_{n-p}^{(p)}(x), \quad .. \quad .. \quad .. \quad (26)$$

where $L_{n-p}^{(p)}(x)$ is the generalized Laguerre polynomial.

The generating function for Angelescus' polynomials was given by Sastri [5] and satisfies the relation

$$\frac{1}{1-t}\exp\left(\frac{-xt}{1-t}\right)\phi\left(\frac{-t}{1-t}\right) = \sum_{n=0}^{\infty}\frac{\Pi_n(x)}{n!}t^n. \quad .. \quad .. \quad (27)$$

It is now clear that the generating function for Angelescus' polynomials is obtained from that of Appell polynomials by replacing t by $\frac{-t}{1-t}$ and then dividing by $(1-t)$. Therefore we have

$$\frac{1}{1-t}\,\phi\left(\frac{-t}{1-t}\right) = \frac{1}{1-t}\int_0^\infty {}_rF_r\left\{\begin{matrix}a_1, a_2, \ldots a_r\\ b_1, b_2, \ldots b_r\end{matrix}; -\frac{ut}{1-t}\right\} d\beta(t). \quad .. \quad (28)$$

Putting $y = -t/(1-t)$ in (27) we have

$$\exp(yx)\,\phi(y) = \sum_{n=0}^{\infty} \frac{\Pi_n(x)}{n!}(-)^n y^n/(1-y)^{n+1}$$

$$= \sum_{r=0}^{\infty} y^r \sum_{n=0}^{r} \frac{\Pi_n(x)}{n!}\,\frac{r!\,(-)^n}{(r-n)!\,n!},$$

Comparing this with (3) we get

$$\frac{P_r(x)}{r!} = \sum_{p=0}^{r} (-)^p \frac{\Pi_p(x)}{p!}\,\frac{1}{p!\,(r-p)!},$$

or

$$P_r(x) = \sum_{p=0}^{r} (-)^p \frac{\Pi_p(x)}{p!}\binom{r}{p}. \quad .. \quad .. \quad .. \quad (29)$$

The corresponding inverse relation can be obtained from (25) by applying Leibnitz theorem and we readily get

$$\frac{\Pi_n(x)}{n!} = \sum_{r=0}^{n} (-)^r \binom{n}{r} P_r(x). \quad .. \quad .. \quad .. \quad (30)$$

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