PRODUCTS OF SUMMABILITY METHODS

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1. In a recent paper Szász (1952) raised the following question.

Let A and B denote any two regular methods of summability for sequences $\{s_n\}$, and let AB denote the iteration-product which associates with a given sequence the A-transform of its B-transform; then does A-summability imply AB-summability ?

In the same paper Szász demonstrated that the answer to this question is in the affirmative when A is Abel and B is (C, α), $\alpha > 0$. *

In the present paper we generalize this result of Szász by replacing (C, α) by the wider class of regular Hausdorff transformations (H, μ), of which Cesàro, Hölder and Euler transformations are well known special cases.

I should like to express my indebtedness to Dr. B. N. Prasad for his kind guidance and encouragement.

2. *Definitions.*

We write

$$f(x) = (1-x)\,\Sigma s_n x^n = \Sigma a_n x^n \; (0 \leqslant x < 1)\,.$$

If the limit

$$\operatorname*{Lt}_{x \to 1-0} f(x) = s$$

exists, then the sequence $\{s_n\}$, or the series Σa_n, is said to be summable (A).

The general Hausdorff transformation (H, μ) is defined by

$$t_m = \sum_n \lambda_{m,\,n}\, s_n,$$

where

$$\lambda_{m,\,n} \begin{cases} = \dbinom{m}{n} \Delta^{m-n} \mu_n & (n \leqslant m), \\[2ex] = 0 & (n > m). \end{cases}$$

* For the case $\alpha = 1$ see Zygmund (1926), p. 189. Incidentally it may be mentioned that the analysis in section 2 of Szász's paper has been vitiated by an obvious oversight in the step just preceding identity (2.4). Indeed (2.4) should read :

$$\frac{1}{y+1} \sum_0^\infty \sigma_n^\alpha \left(1 - \frac{1}{y+1}\right)^n = \frac{\alpha}{y} \int_0^y \left(1 - \frac{u}{y}\right)^{\alpha - 1} \phi(u)\, du,$$

where

$$\phi(u) = f\left(1 - (u+1)^{-1}\right),$$

and, following the notations of Szász's paper, we should finally have

$$A\,(C, \alpha)\,\{s_n\} = (\bar{C}, \alpha)\,\{\phi(u)\}.$$

We shall write

$$\mu_{n,p} = \Delta^p \mu_n,$$

so that

$$\lambda_{m,n} = \binom{m}{n} \mu_{n,m-n} (0 \leqslant n \leqslant m) .$$

If

$$\mu_n = \int_0^1 t^n \, d\chi(t), *$$

where $\chi(t)$ is a real function of bounded variation in $0 \leqslant t \leqslant 1$, then μ_n is called the *moment constant*, of rank n, of $\chi(t)$. We may suppose without loss of generality that

$$\chi(0) = 0 .$$

If, further,

$$\chi(1) = 1,$$

and

$$\chi(+0) = \chi(0) = 0,$$

so that $\chi(t)$ is continuous at $t = 0$, then μ_n is said to be a *regular moment constant*.

We have, in general,

$$\mu_{n,p} = \Delta^p \mu_n = \int_0^1 t^n (1-t)^p \, d\chi(t) .$$

We assume throughout that (H, μ) is a regular Hausdorff transformation.

3. We prove the following theorem.

THEOREM. *If A and B denote two regular methods of summability for sequences $\{s_n\}$, and AB denote their iteration product which associates with a given sequence the A-transform of its B-transform, then A-summability implies AB-summability where A is Abel and B a regular Hausdorff method.*

We require the following lemmas for the proof of our theorem.

LEMMA 1. † *In order that the (H, μ)-transform should be regular it is necessary and sufficient that μ_n should be a regular moment constant.*

LEMMA 2. ‡ *In order that the transformation*

$$G(y) = \int_0^1 g(ty) d\chi(t)$$

should be regular, i.e. that '$g(y) \to s$, as $y \to \infty$' should imply '$G(y) \to s$, as $y \to \infty$', it is necessary and sufficient that $\chi(1) = 1$ and $\chi(+0) = \chi(0) = 0$.

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* The function t^0 is defined at $t = 0$ so as to be continuous. Thus

$$\mu_0 = \int_0^1 d\chi(t) .$$

† Hardy (1949), Theorem 208 (i), p. 260.
‡ Hardy (1949), Theorem 217, p. 276.

Proof of the THEOREM.

Let t_m denote the regular (H, μ)-transform of the sequence $\{s_n\}$. Then our object is to show that if

$$f(x) = (1-x)\sum_n s_n x^n \to s,$$

as $x \to 1-0$, then

$$F(x) = (1-x)\sum_m t_m x^m \to s,$$

as $x \to 1-0$.
We have

$$t_m = \sum_{n=0}^{m} \binom{m}{n} \Delta^{m-n}\mu_n\, s_n$$

$$= \sum_{n=0}^{m} \binom{m}{n} \mu_{n,\, m-n}\, s_n$$

$$= \sum_{n=0}^{m} \binom{m}{n} \left(\int_0^1 t^n(1-t)^{m-n} d\chi(t)\right) s_n$$

$$= \int_0^1 \left(\sum_{n=0}^{m} \binom{m}{n} t^n\,(1-t)^{m-n}\, s_n\right) d\chi(t)\,.$$

Hence

$$\sum_{m=0}^{\infty} t_m x^m = \int_0^1 \left(\sum_{m=0}^{\infty} x^m \sum_{n=0}^{m} \binom{m}{n} t^n\,(1-t)^{m-n}\, s_n\right) d\chi(t)\,.$$

Now, putting

$$I \equiv \sum_{m=0}^{\infty} x^m \sum_{n=0}^{m} \binom{m}{n} t^n\,(1-t)^{m-n}\, s_n,$$

we have

$$I = \sum_{n=0}^{\infty} x^n\, t^n\, s_n \sum_{m \geqslant n} x^{m-n}\,(1-t)^{m-n} \binom{m}{n}$$

$$= \sum_{n=0}^{\infty} x^n\, t^n\, s_n\, \big(1-x\,(1-t)\big)^{-n-1}$$

$$= \big(1-x\,(1-t)\big)^{-1} \left(\sum_{n=0}^{\infty} x^n\, t^n\, \big(1-x\,(1-t)\big)^{-n}\, s_n\right).$$

Thus

$$F(x) = (1-x)\int_0^1 I\, d\chi(t)$$

$$= \int_0^1 \frac{1-x}{1-x(1-t)} \left(\sum_{n=0}^{\infty} \left(\frac{xt}{1-x(1-t)}\right)^n s_n\right) d\chi(t),$$

and, therefore, putting

$$x = 1 - \frac{1}{y+1} = \frac{y}{y+1},$$

we have

$$F\left(\frac{y}{y+1}\right) = \int^{1} \left(1 - \frac{yt}{1+yt}\right) \left(\sum_{n=0}^{\infty} \left(\frac{yt}{1+yt}\right)^{n} s_n \right) d\chi(t)$$

$$= \int^{1} f\left(\frac{yt}{1+yt}\right) d\chi(t).$$

Hence, setting

$$f\left(\frac{y}{1+y}\right) \equiv g(y)$$

and

$$F\left(\frac{y}{1+y}\right) \equiv G(y),$$

we have finally

$$G(y) = \int_{0}^{1} g(yt)\, d\chi(t).$$

Now, as

$$x \to 1-0, \quad y \to \infty,$$

and

$$\underset{y\to\infty}{\mathrm{Lt}}\; g(y) = \underset{x\to 1-0}{\mathrm{Lt}}\; f(x) = s,$$

by hypothesis.

Therefore, by Lemmas 1 and 2,

$$G(y) = F\left(\frac{y}{1+y}\right) \to s, \text{ as } y \to \infty,$$

that is,

$$F(x) \to s, \text{ as } x \to 1-0.$$

This completes the proof of the theorem.

References

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