RADIATION DAMPING IN COMPTON SCATTERING

by T. C. Roy, *Research Fellow, N.I.S.I., Department of Applied Mathematics, University College of Science and Technology, Calcutta*

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The influence of radiation damping in the Compton scattering phenomenon has already been investigated by Wilson (1941) and Power (1945) from the quantum mechanical point of view. One has to tackle Heitler's integral equation in such a case which is extremely difficult to solve exactly. Wilson therefore attempted to solve it after averaging the kernel of the integral equation over the scattering angles. The result that he got by this process showed that the effect of damping is small and can be neglected for all practical purposes in those cases in which the incident photon energy is not unnaturally high. The averaging procedure of Wilson being a very rough approximation, the same problem was again tackled by Power by making a different type of approximation. Her idea is that if the effect of damping be really small then a solution of the above-mentioned integral equation in a series of ascending powers of the fine structure constant may be assumed. By this procedure she calculated the first extra term (1st order approximation term) of the series, the zeroeth order term being the usual Klein Nishina term. This extra term she actually demonstrated to be small, compared to the other one and concluded that the effect of damping confirms Wilson's result.

Recent experience about expansion in powers of the fine structure constant shows that in no way can one assure or ascertain by simple means the convergency of such a procedure. In fact one has to consider the radiative corrections to the Compton scattering which has been considered and discussed in some detail by Feynman (1949). Due to this obvious reason it seems desirable to reconsider the problem yet from another point of view which in our case will be the semivariational procedure of Hsueh and Ma (1945). The reason for doing this is that though approximate this method does not necessitate any further justification on the procedure (as the convergency of the series in the previous case). It therefore seems to be of interest to calculate the scattering cross-section by this procedure and to see to what extent the damping is effective in such a case. The result of our calculation shows that the effect of damping is really small in conformity with the results obtained by Wilson and Power. However, there is a bit of difference. The previous calculations fail to show that the effect of damping is small when the incident photon energy is sufficiently high, owing to the appearance of the factor $\log\left(\frac{k}{\mu}\right)$ in the solution (Wilson, 1941). Though, however, the term does not make its appearance in Power's first order calculation it has not been shown that the same feature will occur in the higher order calculations also. In this respect our calculation shows that the damping effect remains small with any high value of the incident photon energy and even when it is made theoretically infinite. Thus the Klein Nishina formula remains valid even at high energies.

We start with the Heitler's following integral equation and sketch in short the semi-variational procedure of Hsueh and Ma for its solution.

$$U_{\mathbf{k}\mathbf{k}_0} = H_{\mathbf{k}\mathbf{k}_0} + i\pi \sum \int H_{\mathbf{k}\mathbf{k}'} U_{\mathbf{k}'\mathbf{k}_0} \rho_{\mathbf{k}'} d\Omega_{\mathbf{k}'} \qquad \ldots \qquad \ldots \qquad (1)$$

where $\mathbf{k}_0$ corresponds to the initial state, $\mathbf{k}$ the final state, $\mathbf{k}'$ an intermediate state $\rho_{\mathbf{k}'}$ is the number of states per unit energy interval per unit solid angle. The summations integration means really the integration over the whole solid angle and summation over all directions of polarization. Equation (1) can be written in the variational form as follows :—

$$\sum_{k_0 k} \int \delta U^*_{\mathbf{k}_0\mathbf{k}} \left[U_{\mathbf{k}\mathbf{k}_0} - H_{\mathbf{k}\mathbf{k}_0} + i\pi \sum_{k'} \int H_{\mathbf{k}\mathbf{k}'} U_{\mathbf{k}'\mathbf{k}_0} \rho_{\mathbf{k}'} d\Omega_{\mathbf{k}'} \right] \rho_{\mathbf{k}} d\Omega_{\mathbf{k}} = 0. \quad .. \quad (2)$$

Now to obtain an approximate solution of equation (1) we get,

$$U_{\mathbf{k}\mathbf{k}_0} = x\, H_{\mathbf{k}\mathbf{k}_0}$$

where x is a parameter ; equation (2) then becomes,

$$\delta x^* \sum \int H_{\mathbf{k}\mathbf{k}_0} \left[H_{\mathbf{k}_0\mathbf{k}} (x-1) + i\pi x \sum_{\mathbf{k}'} H_{\mathbf{k}\mathbf{k}'}\, H_{\mathbf{k}'\mathbf{k}_0}\, \rho_{\mathbf{k}'} d\Omega_{\mathbf{k}'} \right] \rho_{\mathbf{k}} d\Omega_{\mathbf{k}} = 0 \quad .. \quad (3)$$

which gives

$$x = \frac{\gamma}{\gamma + i\delta}$$

where

$$\gamma = \sum_{kk_0} \int H_{\mathbf{k}_0\mathbf{k}} H_{\mathbf{k}\mathbf{k}_0} \rho_{\mathbf{k}} d\Omega_{\mathbf{k}} \quad .. \quad .. \quad .. \quad .. \quad (4)$$

and

$$\delta = \pi \sum_{kk_0k'} \int H_{\mathbf{k}_0\mathbf{k}} H_{\mathbf{k}\mathbf{k}'} H_{\mathbf{k}'\mathbf{k}_0} \rho_{\mathbf{k}'} \rho_{\mathbf{k}} d\Omega_{\mathbf{k}'} d\Omega_{\mathbf{k}}. \quad .. \quad .. \quad (5)$$

One then easily gets the cross-section of scattering as

$$dQ = dQ_0 \left(1 + \frac{\delta^2}{\gamma^2} \right)^{-1} \quad .. \quad .. \quad .. \quad .. \quad (6)$$

where dQ is the scattering cross-section without damping. Our discussion is therefore mainly focussed on the relative magnitudes of the quantities γ and δ.

To facilitate our calculation we take a Lorentz system in which the total momentum is zero so that the photon momenta $\mathbf{k}$ and $\mathbf{k}_0$ are the same in magnitude but only differ in directions. Also the matrix element $H_{\mathbf{k}\mathbf{k}_0}$ in this high energy region is given by Heitler, 1944

$$H_{\mathbf{k}\mathbf{k}_0} = \frac{\pi e^2}{k^2} A_{\mathbf{k}\mathbf{k}_0} \quad .. \quad .. \quad .. \quad .. \quad (7)$$

where

$$A_{\mathbf{k}\mathbf{k}_0} = U^*_0 \alpha_0 \alpha u + k_0 \frac{u^*_0 \alpha [(\boldsymbol{\alpha}\mathbf{k}_0) + (\boldsymbol{\alpha}\,.\,\mathbf{k})]\, \alpha_0 u}{k_0 E_0 + \mathbf{k}_0\,.\,\mathbf{k}}$$

with

$$\alpha_0 = (\boldsymbol{\alpha} \cdot \mathbf{e}_0)\,, \quad \text{and } \alpha = (\boldsymbol{\alpha}\,.\,\mathbf{e})$$

$\mathbf{e}_0$ and $\mathbf{e}$ being the unit polarization vectors in the $\mathbf{k}_0$ and $\mathbf{k}$ states respectively. The reason for writing E_0 in the denominator is given in Heitler, 1944. U's are as usual normalised Dirac wave functions. One also notes that

$$\rho_{\mathbf{k}} = \tfrac{1}{2} \frac{k^2 d\Omega}{(2\pi)^3}, \quad .. \quad .. \quad .. \quad .. \quad (8)$$

we have all through used $h = c = 1$.

Calculation of γ

To obtain γ from equation (4) one has got to sum over the spin states first. The usual procedure is then to calculate the spur of,

$$\left\{\alpha_0\alpha(E+\beta\mu-\alpha\,.\,\mathbf{k})+\frac{k_0\alpha[(\alpha\,.\,\mathbf{k}_0)+(\alpha\,.\,\mathbf{k})]\alpha_0(E+\beta\mu-\alpha\,.\,\mathbf{k})}{K_0E_0+\mathbf{k}_0\,.\,\mathbf{k}}\right\}\times$$
$$\left\{\alpha\alpha_0(E_0+\beta\mu-\alpha\,.\,\mathbf{k}_0)+\frac{k_0\alpha_0[(\alpha\,.\,\mathbf{k})+(\alpha\,.\,\mathbf{k}_0)]\alpha(E_0+\beta\mu-\alpha\,.\,\mathbf{k}_0)}{K_0E+\mathbf{k}_0\,.\,\mathbf{k}}\right\}$$
$$\begin{aligned}&= \text{sp.}\,\alpha_0\alpha(E+\beta\mu-\alpha\mathbf{k})\alpha\alpha_0(E_0+\beta\mu-\alpha\mathbf{k}_0)\\ &\quad+\lambda^2\,\text{sp.}\,\alpha[\alpha\mathbf{k}_0+\alpha\,.\,\mathbf{k}]\alpha_0(E+\beta\mu-\alpha\,.\,\mathbf{k})\alpha_0[(\alpha\,.\,\mathbf{k})+(\alpha\,.\,\mathbf{k}_0)]\alpha(E_0+\beta\mu-\alpha\,.\,\mathbf{k}_0)\\ &\quad+\lambda\,\text{sp.}\,\alpha_0\alpha(E+\beta\mu-\alpha\mathbf{k})\alpha_0[\alpha\mathbf{k}+\alpha\,.\,\mathbf{k}_0]\alpha(E_0+\beta\mu-\alpha\,.\,\mathbf{k}_0)\\ &\quad+\lambda\,\text{sp.}\,\alpha[\alpha\mathbf{k}+\alpha\mathbf{k}_0]\alpha_0(E+\beta\mu-\alpha.\mathbf{k})\alpha\alpha_0(E_0+\beta\mu-\alpha\mathbf{k}_0)\end{aligned} \quad \ldots \quad (9)$$

where

$$\lambda=\frac{k_0}{k_0E_0+\mathbf{k}_0\,.\,\mathbf{k}}. \quad \ldots \quad (10)$$

On evaluation of the spurs and writing

$$\cos\theta=\frac{(\mathbf{k}_0\mathbf{k})}{k_0{}^2} \quad \ldots \quad (11)$$

(9) reduces to

$$4k_0^2(1+\cos\theta)+16\lambda k_0\,[-k_0^2\cos\theta-4(\mathbf{e}\mathbf{e}_0)(\mathbf{e}\mathbf{k}_0)(\mathbf{e}_0\mathbf{k})+2(\mathbf{e}\,.\,\mathbf{k}_0)^2(\mathbf{e}_0\,.\,\mathbf{k})^2$$
$$+2(\mathbf{e}\,.\,\mathbf{e}_0)^2k_0^2(1+\cos\theta)]. \quad \ldots \quad (12)$$

In the above, and also in the following calculations we have cancelled $(1+\cos\theta)$ with $\frac{1}{k_0\lambda}$, the justification for which is given in the appendix.

Since now we are to sum over all directions of polarization and to integrate over $d\Omega$, we here adopt the following artifice. Let us first take,

$$\mathbf{e}'=\frac{[\mathbf{e}_0\times\mathbf{k}]}{|[\mathbf{e}_0\times\mathbf{k}]|} \quad \ldots \quad (13)$$

and integrate (12) over $d\Omega$; and secondly take,

$$\mathbf{e}''=\frac{|[[\mathbf{e}_0\times\mathbf{k}]\times\mathbf{k}]|}{|[[\mathbf{e}_0\times\mathbf{k}]\times\mathbf{k}]|} \quad \ldots \quad (14)$$

and integrate equation (12) over $d\Omega$, the required result is then the sum of these two integrals. Performing these integrations we get,

$$\gamma=\frac{e^4}{8k_0{}^2}\left[4\log\left(\frac{E_0+k_0}{E_0-k_0}\right)-\tfrac{2}{3}\right] \quad \ldots \quad (15)$$

Calculation of δ

To obtain δ from equation (5) one has again to sum over the spin states. A part of the calculation has already been done by Power which is stated below,

$$\sum_{k'}\int H_{\mathbf{k}\mathbf{k}'}\,H_{\mathbf{k}'\mathbf{k}_0}\,\rho_{\mathbf{k}'}\,d\Omega_{\mathbf{k}'}=-\frac{e^4}{4k_0{}^2}B_{\mathbf{k}\mathbf{k}_0} \quad \ldots \quad (16)$$

where

$$B_{\mathbf{kk}_0} = au_0^* \alpha_0 \alpha u + \frac{b}{k_0}\left[(\mathbf{e}\,.\,\mathbf{k}_0)u_0^*\alpha_0 u + (\mathbf{e}_0\,.\,\mathbf{k})u_0^*\alpha u\right]$$
$$+c(\mathbf{e}_0\,.\,\mathbf{e})u_0^* u + \frac{d}{{k_0}^2}(\mathbf{e}_0\,.\,\mathbf{k})(\mathbf{e}\,.\,\mathbf{k}_0)u_0^* u \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

and

$$a = (1+\cos\theta)L-3,$$
$$b = \frac{2}{1+\cos\theta}\left\{\cos\theta.\,L+\frac{1+3\cos\theta}{1-\cos\theta}\right\},$$
$$c = \frac{2}{1+\cos\theta}\left\{(1-\cos\theta)L+2\right\}$$
$$d = \frac{2}{(1+\cos\theta)^2}\left\{(3\cos\theta-1)L+\frac{8\cos\theta}{1-\cos\theta}\right\} \quad \ldots \quad \ldots \quad (18)$$

where

$$L = \frac{2}{1+\cos\theta}\log\frac{1-\cos\theta}{2}. \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

Hence in order to calculate δ from equation (5), one uses equations (7) and (16) and calculates the sum over the spin states of,

$$A_{\mathbf{kk}_0}\, B_{\mathbf{k}_0\mathbf{k}}$$

which is of the form,

$$\frac{1}{8E_0E}\left(Aa+Bb+Cc+Dd\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

where A, B, C, D are to be evaluated, they being,

$$A = \text{sp.}\ \{\alpha_0\alpha(E+\beta\mu-\boldsymbol{\alpha}\,.\,\mathbf{k})\alpha\alpha_0(E_0+\beta\mu-\boldsymbol{\alpha}\,.\,\mathbf{k}_0)+\lambda\alpha_0\alpha(E+\beta\mu-\boldsymbol{\alpha}\,.\,\mathbf{k})\times$$
$$\alpha_0(\boldsymbol{\alpha}\mathbf{k}+\boldsymbol{\alpha}\mathbf{k}_0)\alpha(E_0+\beta\mu-\boldsymbol{\alpha}\,.\,\mathbf{k}_0)\} \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

$$B = \frac{(ek_0)}{k_0}\text{sp.}\left\{\alpha_0(E+\beta\mu-\boldsymbol{\alpha}\mathbf{k})\alpha\alpha_0(E_0+\beta\mu-\boldsymbol{\alpha}\mathbf{k}_0)+\lambda\alpha_0(E+\beta\mu-\boldsymbol{\alpha}\mathbf{k})\right.$$
$$\left.\times\,\alpha_0(\boldsymbol{\alpha}\mathbf{k}+\boldsymbol{\alpha}\mathbf{k}_0)\alpha(E_0+\beta\mu-\boldsymbol{\alpha}\mathbf{k}_0)\right\}+\frac{(\mathbf{e}_0\mathbf{k})}{k_0}\text{sp}\left\{\alpha(E+\beta\mu-\boldsymbol{\alpha}\,.\,\mathbf{k})\alpha\alpha_0(E_0+\beta\mu-\boldsymbol{\alpha}\mathbf{k}_0)\right.$$
$$\left.+\lambda\alpha(E+\beta\mu-\boldsymbol{\alpha}\mathbf{k})\alpha_0(\boldsymbol{\alpha}\mathbf{k}+\boldsymbol{\alpha}\mathbf{k}_0)\alpha(E_0+\beta\mu-\boldsymbol{\alpha}\mathbf{k}_0)\right\} \quad \ldots \quad \ldots \quad (22)$$

$$\frac{C}{(\mathbf{e}_0.\mathbf{e})} = \frac{D}{\dfrac{(\mathbf{e}_0\mathbf{k})(\mathbf{e}\mathbf{k}_0)}{{k_0}^2}} = \text{sp}\left\{(E+\beta\mu-\boldsymbol{\alpha}\mathbf{k})\alpha\alpha_0(E_0+\beta\mu-\alpha\mathbf{k}_0)+\lambda(E+\beta\mu-\boldsymbol{\alpha}\mathbf{k})\right.$$
$$\left.\times\,\alpha_0(\boldsymbol{\alpha}\mathbf{k}+\boldsymbol{\alpha}\mathbf{k}_0)\alpha(E_0+\beta\mu-\boldsymbol{\alpha}\mathbf{k}_0)\right\}. \quad (23)$$

Calculating the spurs we get,

$$4A = E_0^2+\mu^2+(\mathbf{kk}_0)-\lambda E_0\ \{2(\mathbf{kk}_0)+2k_0^2-4(\mathbf{ee}_0)^2(\mathbf{kk}_0)$$
$$+4(\mathbf{ee}_0)(\mathbf{e}_0\mathbf{k})(\mathbf{ek}_0)-4k_0^2(\mathbf{ee}_0)^2-2(\mathbf{ek}_0)^2-2(\mathbf{e}_0\mathbf{k})^2\} \quad \ldots \quad \ldots \quad (24)$$

$$\frac{16B}{k_0} = \frac{E_0}{4}\left\{(\mathbf{ek}_0)^2+(\mathbf{e}_0\mathbf{k})^2\right\}+\lambda(\mathbf{e}_0\mathbf{k})^2(\mathbf{ek}_0)^2-\frac{\lambda}{2}(\mathbf{kk}_0)\left\{(\mathbf{ek}_0)^2+(\mathbf{e}_0\mathbf{k})^2\right\}$$
$$-\lambda k_0^2(\mathbf{ee}_0)(\mathbf{e}_0\mathbf{k})(\mathbf{ek}_0)-\lambda(\mathbf{ee}_0)(\mathbf{kk}_0)(\mathbf{e}_0\mathbf{k})(\mathbf{ek}_0) \quad \ldots \quad \ldots \quad (25)$$

$$\frac{4C}{(\mathbf{e}\,.\,\mathbf{e}_0)} = \frac{4D}{\frac{(\mathbf{e}_0\mathbf{k})(\mathbf{ek}_0)}{k_0^2}} = (\mathbf{e}.\mathbf{e}_0)(E_0^2+\mu^2)+(\mathbf{e}\,.\,\mathbf{e}_0)(\mathbf{k}\,.\,\mathbf{k}_0)-(\mathbf{e}_0\mathbf{k})(\mathbf{ek}_0)$$
$$+2\lambda E_0[(\mathbf{ee}_0)(\mathbf{kk}_0)+(\mathbf{ee}_0)k_0^2-(\mathbf{ek}_0)(\mathbf{e}_0\mathbf{k})]. \quad \ldots \quad (26)$$

Equation (20) is now therefore completely known. We now proceed to integrate over the solid angle $d\Omega$ and sum over all directions of polarization which we do just in the same way as in the case of evaluation of γ. We therefore integrate using once,

$$\mathbf{e}' = \frac{[\mathbf{e}_0\times\mathbf{k}]}{|\,\mathbf{e}_0\times\mathbf{k}\,|}$$

and then

$$\mathbf{e}'' = \frac{[[\mathbf{e}_0\times\mathbf{k}]\times\mathbf{k}]}{|\,[\mathbf{e}_0\times\mathbf{k}]\times\mathbf{k}\,|}$$

and sum the two results. Performing these integrations we finally get,

$$\delta = \frac{e^6}{32k_0^2}\left[\frac{7\pi^2}{12}+\frac{296}{9}-5\log\left(\frac{E_0+k_0}{E_0-k_0}\right)\right]. \quad \ldots \quad \ldots \quad (27)$$

From equation (6) we see that the quantity $\frac{\delta^2}{\gamma^2}$ measures the effect of damping. We are now in a position to evaluate this quantity with the help of equations (15) and (27) for different values of the incident photon energy. The behaviour of this quantity is shown in the following table for all large values of $\mathbf{k}_0$ for which the approximation is valid and in which case the effect of damping is not at all important.

TABLE

k_0/μ	137	137^2	137^3	∞
δ^2/γ^2	$1.4.10^{-6}$	$2.6.10^{-7}$	$1.2.10^{-6}$	$5.2.10^{-6}$

We therefore see that for all values of the incident photon energies the effect of radiation damping is entirely negligible.

CONCLUSION

The power series treatment in the fine structure constant of the Heitler's integral equation does not allow one to conclude that the effect of radiation damping is really small at all energies owing to the appearance of the factors $\log\left(\frac{k}{\mu}\right)$ in the terms which not only increase with k but may make the series expansion divergent, and it is not even possible to find an upper bound of k for which the series remains convergent. Though the situation was thought to be understandable from the 1st order term, calculated by Power which fortunately did not contain any such log $\left(\frac{k}{\mu}\right)$ factor, the calculation does not enable one to understand the nature of the higher order terms. In fact the convergency of the series cannot be tested for any value of k, unless one gets an idea of its general term. The semi-variational treatment of the integral equation, however, states that even for any arbitrarily large values of k the effect of damping is really small.

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Appendix

Let us consider an expression of the form

$$\frac{(1+\cos\theta)}{1+\cos\theta+\dfrac{\mu^2}{2k_0^2}}$$

we can write it as

$$1-\frac{\mu^2/2k_0}{1+\cos\theta+\mu^2/2k_0^2}$$

now (from the very start) we have neglected quantities of the order of μ^2/k_0^2 in the numerator (Heitler) and hence we write,

$$\frac{1+\cos\theta}{1+\cos\theta+\mu^2/2k_0^2}\simeq 1$$

However, we can show that even when the part neglected above be taken into account its integral will again go to zero as μ^2/k_0^2 is made zero as it should be. This is as follows :—

$$\int_{-1}^{1}\left\{\frac{\mu^2}{2k_0^2}\Big/\left(1+\cos\theta+\mu^2/2k_0^2\right)\right\}d(\cos\theta)=\frac{\mu^2}{2k_0^2}\left[\log\left(\frac{2+\mu^2/2k_0^2}{\mu^2/2k_0^2}\right)\right]$$

and this $\rightarrow 0$ as $\dfrac{\mu^2}{2k_0^2}\rightarrow 0$.

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