EFFECT OF E.M. RADIATION ON THE SELF-ENERGY OF FREE-ELECTRONS. I.

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I. A unique feature of quantum electrodynamics is the zero-point fluctuations of the electro-magnetic field. An electron, in vacuum, is in constant interaction with these oscillations of infinite energy, and the result is, an increase in the self-energy of the electron. This interaction energy is usually called the Transverse Self-Energy of the Electron. What happens may be visualised thus:

The fluctuations induce a current in the vacuum, as does the virtual photon emission. But, as the current due to virtual photon emission interacts with the emitted virtual field, so the current due to the zero-point amplitude interacts with the amplitude itself (Sawada, 1949). The transverse interaction energy has been calculated by many authors (Weisskopf, 1939) and comes out infinite both on Dirac's one-electron theory, and on 'hole' theory, though in reality (so we suppose) it is a small effect. This assumption is very necessary, in that one cannot restore to the renormalisation procedure, without it, which in turn, has proved of immense use, in interpreting the Lamb Shift (Bethe, 1947) and the anomalous magnetic moment of the electron (Schwinger, 1948). The apparent divergence is ascribed to improper handling of the electro-magnetic mass by present theories which seem to suffer from some deep-seated limitation. In fact, the objectionable features of quantum electrodynamics are much too obvious in all processes which involve virtual transitions in the ultraviolet.

However, there arise additional contributions to the electronic self-energy, if the electron happens to be situated in a transverse radiation field (Auluck and Kothari, 1953, I. Singh, 1953) or is surrounded by other electrons (Salam, 1953). Here we will confine ourselves to the case of a free-electron in an atmosphere of 'free photons'. Due to interaction with the transverse radiation field, the electron is perturbed to undergo transitions involving real photons, which on Dirac's one-electron theory are induced in two ways: the electron in state, $|n)$, in interaction with an assembly of $N(k)$ quanta, of momentum $\vec{k}$ and of energy $|\vec{k}|$ (the velocity of light is taken as unity) may jump to the intermediate state, $|i)$, by either the emission of a photon of momentum $\vec{k}$ or by the absorption of a surrounding photon. In the first case, there will be two more photons per oscillator of the radiation field than there will be in the second case. The interaction energy, arising due to such transitions, is given by the second order matrix element,

$$W = \sum_i \frac{(n|H'|i)\,(i|H'|n)}{E_n - E_i} \qquad \ldots \quad \ldots \quad \ldots \quad (1)$$

where the summation is over all intermediate states. H', is the contribution, due to interaction between the two subsystems, to the total Hamiltonian of the system consisting of the electron and the radiation field, and is given by—

$$H' = -\sum e\left(\vec{\alpha}\,.\,\vec{A}\right) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where, e is the unit electronic charge, $\vec{\alpha}$ is the Dirac vector matrix $(\alpha_x, \alpha_y, \alpha_z)$ and $\vec{A}$ is the vector potential of the external transverse field, given by

$$A = \sum_\lambda \vec{e}_\lambda \, (A_\lambda/\sqrt{2}) \left(c_\lambda \exp{(i\, k_\lambda x)} + c_\lambda^* \exp{(-ik_\lambda \, . \, x)}\right)$$

with $$A_\lambda = (4\pi/k_\lambda \, . \, V)^{\frac{1}{2}}. \quad \ldots \quad (3)$$

V, is the normalisation volume, and c_λ^*, c_λ are respectively the creation and destruction operators of a photon, in a state with momentum $\vec{k}_\lambda$ and polarisation vector, $\vec{e}_\lambda$ such that,

$$\left. \begin{aligned} \alpha_\lambda &\equiv \vec{e}_\lambda \, . \, \vec{\alpha} \\ \text{and} \qquad c_\lambda c_{\lambda'}^* - c_\lambda^* c_{\lambda'} &= \delta_{\lambda\lambda'}, \text{ etc.} \end{aligned} \right\} \quad \ldots \quad (4)$$

since both emission and absorption processes are possible, we have, instead of (1),

$$W = 2\pi e^2 \hbar^2 \sum_i \left\{ \frac{N(k)+1}{k} \frac{(n \,|\, \alpha \,|\, i_1)\,(i_1 \,|\, \alpha \,|\, n)}{E_n - E_{i1}} + \frac{N(k)}{k} \frac{(n \,|\, \alpha \,|\, i2)\,(i2 \,|\, \alpha \,|\, n)}{E_n - E_{i2}} \right\} \quad \ldots \quad (5)$$

where the sum is also over the two directions of polarisation of the quanta.

Summing over spin directions of the initial and final state of the electron, and over the two possible kinds of energy in the intermediate state, we have for a free electron at rest, in an atmosphere of free photons,

$$\begin{aligned} W = 2\pi e^2 \hbar^2 \sum \Big\{ & \frac{N(k)+1}{k} \cdot \frac{1}{2} \cdot \text{sp} \, \frac{(\mu - k)\,(H+E) + \alpha H' \alpha \,(H+E)}{-4\mu E} \\ & + \frac{N(k)}{k} \cdot \frac{1}{2} \cdot \text{sp} \, \frac{(\mu + k)\,(H+E) + \alpha H'' \alpha \,(H+E)}{4\mu \, E} \Big\} \quad \ldots \quad (6) \end{aligned}$$

$$\left. \begin{aligned} \text{where,} \qquad H+E &\equiv \mu\,(1+\beta) \\ H' &\equiv \beta\mu + (\vec{\alpha}\,\vec{p}') \\ H'' &\equiv \beta\mu + (\vec{\alpha}\,\vec{p}'') \\ \text{with,} \qquad \vec{p}' &= -\vec{k} \\ \text{and} \qquad \vec{p}'' &= +\vec{k}. \end{aligned} \right\} \quad \ldots \quad (7)$$

Here β is another Dirac matrix such that

$$\alpha_i \beta + \beta \alpha_i = 0, \qquad (i = x, y, z),$$

and μ is the mechanical mass of the electron. Assuming the photons to obey black-body distribution,

$$N(k) = \frac{1}{\exp{(k/RT)} - 1} \quad \ldots \quad (8)$$

where R is the Boltzmann gas constant, and T is the temperature of the photon gas.

A summation over all quanta in expression (6) leads to an infinite result. But, this divergence is due to the presence of the usual transverse (vacuum) self-energy term, in the said expression. This term will arise even when there is a complete absence of photons. Subtracting therefore, from eq. (6), this term, we get for the self-energy due to photons,

$$W = \frac{2\alpha}{\pi\mu}(RT)^2 . \xi(1, 1) \quad .. \quad .. \quad .. \quad .. \quad (9)$$

where

$$\xi(s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} \exp(-ax)}{1 - \exp(-x)} dx \quad .. \quad .. \quad .. \quad (10)$$

is the well-known Rieman Zeta function. When one substitutes in expression (9), the value for the first Zeta function, $\xi(1, 1)$, one gets,

$$W' = \frac{\pi}{3} . \alpha . \frac{RT}{\mu} . RT \quad .. \quad .. \quad .. \quad .. \quad (11)$$

for the additional interaction energy due to the presence of a transverse field, obeying Planck's distribution. Here, α is the Sommerfeld fine-structure constant.

II. The above calculations have been made on Dirac's One-electron theory, wherein the electron can find itself even in the physically unacceptable states of negative mass. On the positron theory, however, such states in general, are all supposed to be occupied by electrons, thus preventing the positive states from the otherwise inevitable decay. Transitions to these occupied negative states are forbidden by Pauli Exclusion Principle.

Because of its special (charged) structure, the vacuum, on hole theory, has important physical properties, which have a decisive bearing on the self-energy problem of electrons. The vacuum is characterised by an infinite charge density, and behaves in many respects, like an intense electro-magnetic field. An electron placed in the vacuum, will obviously deform the latent electron distribution. A hole is created at the position of the electron, and the electron-charge is effectively spread, in that it can be located up to distances comparable to the Compton Wavelength, h/μ. At a distance, ξ, the charge-density is governed by the Weiskoff relation (with a quadratic singularity at $\xi = 0$)

$$G(\xi) = \frac{ie^2}{2\pi} \cdot \frac{\mu}{h} \cdot \frac{1}{\xi} \frac{\partial}{\partial \xi} H_0^{(1)}(i\mu\xi/h), \quad .. \quad .. \quad .. \quad (12)$$

where $H_0^{(1)}(x)$ is a Hankel function of the first kind. The vacuum polarisation increases the effective charge of the electron by a small amount. The interaction between the electron and the oscillations of the field is also modified by the presence of the latent electron pairs in the vacuum, and the perturbation is greater for the higher (compared to the minimum frequency of pair-creation, $2\mu/h$) frequencies of the field. The expression for the transverse (vacuum) self-energy is made more convergent, in that the infinity changes from a quadratic to one, which is of a logarithmic order, only.

The self-energy of an electron in a state $|n)$ on positron theory is the 'self-energy of one electron in state $|n)$ plus the vacuum electrons', minus the 'self-energy of the vacuum electrons alone', or

$$W = W_{vac+1} - W_{vac}. \quad .. \quad .. \quad .. \quad .. \quad (13)$$

In considering the influence of electro-magnetic radiation on the transverse self-energy of a free electron at rest, on Dirac's One-electron theory, we had to deal with four intermediate states, two of them being of negative energy. On the 'hole' theory, this number will be maintained, but instead of the two negative-energy intermediate states in which the electron has momentum, $\pm \vec{k}$, we have now, two new states in which the latent electrons take part. The vacuum electron of momentum, $\vec{p}-\vec{k}$, absorbs a photon from the radiation field and goes over to the state, $\vec{p}$ of the original electron, which on the other hand jumps into the hole so created, emitting a photon, in the process. Thus in the intermediate state we have an electron-positron pair, and in the annihilation process, it is not this electron which takes part, but the electron with which we started. The positive state decays and is recreated by an electron supplied from the vacuum. The second intermediate state arises by the emission of a photon, $\vec{k}$, by a vacuum electron of momentum $\vec{p}+\vec{k}$ which goes over to p, while the original electron fills the hole, by absorbing a photon $\vec{k}$. Thus, we have, using eq. (13),

$$W = 2\pi e^2\hbar^2 \sum_{r,\,v} \left[\frac{N(k)+1}{k}\left\{\frac{(n\,|\,\alpha\,|\,r)(r\,|\,\alpha\,|\,n)}{E_n-k-E_r}+\frac{(n\,|\,\alpha\,|\,v)(v\,|\,\alpha\,|\,n)}{E_n+k+E_v}\right\}\right.$$
$$\left.+\frac{N(k)}{k}\left\{\frac{(n\,|\,\alpha\,|\,r)(r\,|\,\alpha\,|\,n)}{E_n+k+E_r}+\frac{(n\,|\,\alpha\,|\,v)(v\,|\,\alpha\,|\,n)}{E_n-k-E_v}\right\}\right]. \quad \ldots \quad (14)$$

There is no need to consider processes involving vacuum electrons explicitly. They are automatically taken account of, if one makes use of the familiar relation

$$(n_1\,|\,O\,|\,n_1)+(n_2\,|\,O\,|\,n_2)-(n_3\,|\,O\,|\,n_3)-(n_4\,|\,O\,|\,n_4) = \frac{1}{E}\,\mathrm{sp}\,O.H \quad \ldots \quad (15)$$

where $|\,n_1)$ and $|\,n_2)$ are the Eigen-functions of the Dirac equation, corresponding to the positive energy states and $|\,n_3)$ and $|\,n_4)$, to the remaining two states of negative energy. Proceeding in this way, we get,

$$W = 2\pi e^2\hbar^2 \sum \left\{\frac{N(k)+1}{k^2}\cdot\frac{1}{2}\cdot\mathrm{sp}\,\frac{\alpha(H'+E')\alpha(E'+k+H)H}{-4EE'(k+E')}\right.$$
$$\left.+\frac{N(k)}{k^2}\cdot\frac{1}{2}\,\mathrm{sp}\,\frac{\alpha(H''+E'')\alpha(E''-k+H)H}{4EE''(k-E'')}\right\}. \quad \ldots \quad (16)$$

Again, separating out the transverse (vacuum) self-energy term and using expression (8), we have—

$$W' = \frac{\pi}{3}\,.\,\alpha\,.\,\frac{RT}{\mu}\,.\,RT$$

as before. Thus, the calculations on Dirac's One-electron theory and on 'hole' theory lead to the same result. This result could be anticipated, because the process is symmetric, in that we have both emission and absorption transitions of the original electron, and also because the wave-functions are antisymmetric in the co-ordinates of the initial and the final electrons, on 'hole' theory.

It is interesting, that while the usual transverse self-energy of an electron (absence of all quanta) comes out different on Dirac's One-electron theory and on 'hole' theory, this is not the case for the additional effects arising due to the presence of an external transverse radiation field, which are, so to say, independent of the approach employed.

III. So far, we have tacitly assumed, that the electron under consideration has no momentum. The more general case is to associate a momentum $\vec{p}$ with the electron so that

$$E^2 = p^2+\mu^2. \qquad \ldots \quad (17)$$

After making this change in expression (5), we have to carry out the summations exactly as before. Therefore, we shall not enter into the details of this straightforward calculation, but shall only quote the final expression for the self-energy of the electron of energy E due to the presence of external radiation. This is

$$W' = \frac{\pi}{3} \,.\, \alpha \,.\, \frac{RT}{E} \,.\, RT \qquad \ldots \quad (18)$$

which is identical with eq. (11) for $p = 0$. In both absorption and emission, we get a term,

$$\left(\frac{\mu^2}{2p} \log \frac{(E+p)}{(E-p)} - E\right) \int N(k)dk \qquad \ldots \quad (19)$$

which, obviously suffers from an infra-red catastrophe. However, the two processes lead to terms which are opposite in sign, and so the divergent elements disappear from the final expression (18) which, therefore, is finite.

In investigating the effect of radiation on the self-energy problem, we have so far assumed that the photon distribution obeys Planck's law. We can treat however, a very general case, by assuming that the intensity of the radiation of frequency $|\vec{k}|$ is some unspecified function of $|\vec{k}|$ and the radiation is enclosed in a volume G which also contains the electron under consideration. Then

$$N(k) = G \,.\, \frac{I(k)}{k} \qquad \ldots \quad (20)$$

and (18) is given by

$$W' = \frac{2\alpha \,.\, G}{\pi E} \int I(k)dk. \qquad \ldots \quad (21)$$

The limits of integration are determined by the particular range of quanta in loose interaction with the electron. Expression (21) shows, that the transverse self-energy is the same for two different frequencies of the photons, so long as their respective intensities are equal in the radiation bath.

The problem of a bound electron will be treated in a subsequent paper.

IV. In conclusion, it is indeed, a pleasure to thank Dr. F. C. Auluck for valuable guidance, and Professor D. S. Kothari for his kind interest and stimulating discussions. Last, but not the least, my thanks are due to the Atomic Energy Commission, Government of India, for the award of a Fellowship.

Summary

The change in self-energy of an electron due to the presence of an electro-magnetic field is investigated. It is shown that this contribution is finite and comes out the same on Dirac's One-electron theory (all negative energy states empty) and on 'hole' theory (all negative states occupied).

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