THE EFFECT OF VORTICITY ON THE SPECIFIC HEAT RATIO OF A NON-RELATIVISTIC FERMI-DIRAC GAS

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Beeton (1950) has recently investigated the effect of vorticity on the pressure-volume index at different adiabatic compression and expansion ratios in a classical gas. In an actual fluid there will be generally a number of vortices interacting with each other and therefore the general problem is difficult to formulate. To avoid this difficulty Beeton has considered the case of only one cylindrical vortex present in the fluid. He assumes that initially total energy (enthalpy and kinetic energy) is constant throughout the vortex. He has neglected the effect of viscosity. He has shown that the pressure-volume index increases with vorticity. We have generalised the above result in the case of non-relativistic Fermi-Dirac gas taking degeneracy into account.

Consider an assembly of N free electrons occupying a volume V at temperature T. The energy E, pressure P and entropy S are given for a non-relativistic Fermi-Dirac gas by the following relations (Kothari and Singh, 1942):

$$E = \frac{2\pi gN}{h^3 n}(2mkT)^{\frac{3}{2}} kT\, F_{\frac{3}{2}}(A), \quad \ldots \quad (1)$$

$$P = \frac{2\pi g}{3h^3}(2mkT)^{\frac{3}{2}} kT\, F_{\frac{3}{2}}(A), \quad \ldots \quad (2)$$

and

$$S = \frac{4\pi gN}{h^3 n}(2mkT)^{\frac{3}{2}} k\left\{\frac{5}{2}F_{\frac{3}{2}}(A) + \frac{3}{2}\, ln\, A.\, F_{\frac{1}{2}}(A)\right\}. \quad \ldots \quad (3)$$

Here,

m = rest mass of the electron,
k = Boltzmann constant,
g = Weight factor,
n = concentration = N/V,
A = degeneracy parameter = $nh^3/g(2\pi mkT)^{\frac{3}{2}}$,
h = Planck's constant,
$F_\beta(A)$ = Fermi-Dirac function corresponding to index β

$$= \int_0^\infty \frac{x^\beta}{\frac{1}{A}e^x + 1}\, dx = \text{a function of } A$$

where $x = \epsilon/kT$ and ϵ = kinetic energy of a particle.

We can put the entropy in the form

$$S = \frac{4Ngk}{\sqrt{\pi}A}\left\{\frac{5}{2}F_{\frac{3}{2}}(A) + \frac{3}{2}\, ln\, A.\, F_{\frac{1}{2}}(A)\right\} \quad \ldots \quad (4)$$

An adiabatic change is an isentropic process. Hence, from equation (3) or (4), it is evident that $T^{\frac{3}{2}}/n$ or A remains constant for an adiabatic process. The relations (1) and (2) then reduce to

$$E = \tfrac{3}{2}\, CNkT, \qquad (5)$$

and

$$P = C\rho NkT \qquad (6)$$

where ρ is the density. Here $C = 4F_{\frac{3}{2}}(A)/3\pi^{\frac{1}{2}}A$, A being a constant quantity.

The above relations (5) and (6) differ from those given for a classical gas by the constant C.

Following Beeton, we assume initially that total energy for every element remains constant throughout the vortex. Then using suffix (0) for initial conditions, we have, for an element of the vortex moving with the velocity v_0,

$$\frac{v_0^2}{2} + E_0 + P_0 V_0 = (\text{constant})_0,$$

or

$$v_0^2 = 5\, CNK(T_{t_0} - T_0) \qquad (7)$$

where $\frac{5}{2}\, CNkT_{t_0} = (\text{constant})_0$. Here we will call T_{t_0} the initial total temperature. Further we define the total pressure P_t in this case as

$$\left(\frac{T_{t_0}}{T_0}\right) = \left(\frac{P_{t_0}}{P_0}\right)^{\frac{2}{5}}. \qquad (8)$$

For an element of gas moving with a velocity v at a radius r of the cylindrical vortex, we have for equilibrium,

$$\frac{v^2}{r}\rho\, dr = dP \qquad (9)$$

or for initial conditions,

$$\frac{v_0^2}{r_0}\rho_0 dr_0 = dP_0. \qquad (10)$$

Eliminating v_0^2 from equations (7) and (10) and using equation (8), we get,

$$5\frac{dr_0}{r_0} = \frac{dP_0}{P_0\left\{\left(\frac{P_{t_0}}{P_0}\right)^{\frac{2}{5}} - 1\right\}}. \qquad (11)$$

If $r_0 = r_0'$ and $P_0 = P_0'$ at the outer periphery of the vortex, then for the initial state the integral gives,

$$\left(\frac{r_0}{r_0'}\right)^2 = \frac{1 - \left(\frac{P_0'}{P_{t_0}}\right)^{\frac{2}{5}}}{1 - \left(\frac{P_0}{P_{t_0}}\right)^{\frac{2}{5}}} \qquad (11)$$

As r decreases towards the centre of the vortex, the pressure also decreases. At some radius $\bar{r}$ the pressure has become zero, therefore,

$$\left(\frac{\bar{r}_0}{r_0'}\right)^2 = 1 - \left(\frac{P_0'}{P_{t_0}}\right)^{\frac{2}{5}} = 1 - \frac{T_0'}{T_{t_0}}, \qquad (12)$$

and hence

$$\left(\frac{\bar{r}_0}{r_0}\right)^2 = 1-\left(\frac{P_0}{P_{t_0}}\right)^{\frac{2}{5}} = 1-\frac{T_0}{T_{t_0}}. \qquad \ldots \quad \ldots \quad \ldots \quad (13)$$

We have also from equation (7)

$$\frac{v_0^2}{5\,CNk} = T_{t_0}\left(\frac{\bar{r}_0}{r_0}\right)^2. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

Hence the angular momentum is everywhere the same. For the sake of simplicity, we assume that there are no viscous forces. Hence the angular momentum of every element remains constant no matter what happens to the vortex as a whole in the way of expansion or compression. Therefore the condition that the product (vr) is constant throughout the vortex, which has been shown above to hold initially, will remain true during subsequent changes. Therefore, we have,

$$v^2r^2 = 5\,CNk\,T_{t_0}\,\bar{r}_0^2. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

Assuming the change in a particular element of the vortex to be reversible, we have,

$$\frac{T}{T_0} = \left(\frac{P_0}{P}\right)^{\frac{2}{5}}.$$

But by the definition of total temperature and pressure,

$$\frac{T_0}{T_{t_0}} = \left(\frac{P_0}{P_{t_0}}\right)^{\frac{2}{5}}.$$

Hence

$$\frac{T}{T_{t_0}} = \left(\frac{P}{P_{t_0}}\right)^{\frac{2}{5}}$$

i.e. the temperature and pressure at all points of the vortex and at any instant are related by,

$$\frac{T}{P^{\frac{2}{5}}} = \text{constant}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

At any instant, therefore, we have,

$$\frac{dT}{dP} = \frac{2}{5}\frac{T}{P}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

Now we will show that our earlier assumption that initially the total energy and hence the total temperature are the same for all points holds good for all subsequent states after expansion or compression by external forces.

We get, from equations (6), (9) and (15),

$$\frac{dP}{dr} = 5T_{t_0}\bar{r}_0\,\frac{P}{r^3T}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

Now with equations (17) and (18),

$$\frac{dT}{dr} = \frac{dT}{dP}\cdot\frac{dP}{dr} = 2T_{t_0}\,\bar{r}_0\,\frac{1}{r^3}. \qquad \ldots \quad \ldots \quad \ldots \quad (19)$$

Integrating the above equation, and taking $T = T'$ for $r = r'$, we get,

$$T = T' - T_{t_0} \bar{r}_0^2 \left(\frac{1}{r^2} - \frac{1}{r'^2}\right) \quad \ldots \quad (20)$$

Equation (20) thus gives temperature as a function of radius.

Putting $r = r'$ for $T = 0$, we get,

$$T' = T_{t_0} \bar{r}_0^2 \left(\frac{1}{\bar{r}^2} - \frac{1}{r'^2}\right). \quad \ldots \quad (21)$$

Hence,

$$\left(\frac{T}{T'}\right) = \left(\frac{\frac{1}{\bar{r}^2} - \frac{1}{r^2}}{\frac{1}{\bar{r}^2} - \frac{1}{r'^2}}\right). \quad \ldots \quad (22)$$

Therefore

$$T_t = T + \frac{v^2}{5NCk}$$

$$= T_{t_0} \frac{\bar{r}_0^2}{\bar{r}^2} = \text{constant for all elements of the vortex.} \quad \ldots \quad (23)$$

The mass of an element of the vortex is given by

$$\delta m = \rho\,(2\pi r\, dr)$$

$$= 2\pi\rho' \left(\frac{\frac{1}{\bar{r}^2} - \frac{1}{r^2}}{\frac{1}{\bar{r}^2} - \frac{1}{r'^2}}\right)^{\frac{3}{2}} r\,\delta r,$$

as $\rho \propto T^{\frac{3}{2}}$ for a non-relativistic Fermi-Dirac gas.

Therefore,

$$\frac{m}{2\pi\rho'} = \int_{\bar{r}}^{r'} \left(\frac{\frac{1}{\bar{r}^2} - \frac{1}{r^2}}{\frac{1}{\bar{r}^2} - \frac{1}{r'^2}}\right)^{\frac{3}{2}} r\, dr. \quad \ldots \quad (24)$$

Putting

$$\left(\frac{1}{\bar{r}^2} - \frac{1}{r^2}\right) = Z\left(\frac{1}{\bar{r}^2} - \frac{1}{r'^2}\right),$$

and t = vorticity parameter $= \dfrac{r'^2}{r'^2 - \bar{r}^2} = \dfrac{T_t}{T'}$,

equation (24) reduces to the simple form,

$$\frac{m}{2\pi\rho'} = \frac{r'^2}{2t}\left(1 - \frac{1}{t}\right) \int_0^1 Z^{\frac{3}{2}} \left\{1 - \frac{Z}{t}\right\}^{-2} dZ,$$

$$= r'^2 f(t), \quad \ldots \quad (25)$$

where

$$f(t) = \frac{1}{2t}\left(1-\frac{1}{t}\right)\int_0^1 Z^{\frac{3}{2}}\left\{1-\frac{Z}{t}\right\}^{-2} dZ,$$

$$= \frac{3}{2}t-1-\frac{3}{4}\sqrt{t}\,(t-1)\,ln\,\frac{\sqrt{t}+1}{\sqrt{t}-1}. \quad \text{.. .. .. (26)}$$

Since $\rho \propto P^{\frac{3}{5}}$ for a Fermi-Dirac gas, we have by the conservation of mass,

$$\left(\frac{P'}{P_0'}\right)^{\frac{3}{5}}\left(\frac{r'^2}{r_0'^2}\right)\left(\frac{f(t)}{f(t_0)}\right) = 1. \quad \text{.. .. .. .. (27)}$$

Further

$$t = \frac{T_t}{T'} = 1+\frac{T_{t_0}}{T'}\frac{\bar{r}_0^2}{r'^2}.$$

$$t_0 = 1+\frac{T_{t_0}}{T_0'}\frac{\bar{r}_0^2}{r_0'^2}.$$

Therefore,

$$\frac{t-1}{t_0-1} = \frac{T_0'}{T'}\frac{r_0'^2}{r'^2}.$$

Hence

$$\left(\frac{t-1}{t_0-1}\right)\left(\frac{P'}{P_0'}\right)^{\frac{2}{5}}\left(\frac{r'^2}{r_0'^2}\right) = 1. \quad \text{.. .. .. (28)}$$

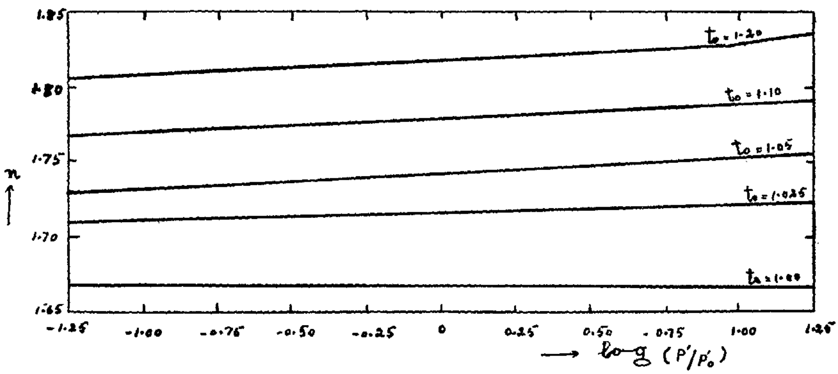
Therefore, equations (27) and (28) together give,

$$\left(\frac{P'}{P_0'}\right)^{\frac{1}{5}}\left(\frac{f(t)}{f(t_0)}\right) = \frac{t-1}{t_0-1}. \quad \text{.. .. .. .. (29)}$$

Since $f(t)$ is a known function, equations (28) and (29) are enough to determine the variation of volume ratio $\left(\frac{r'^2}{r_0'^2}\right)$ with the applied pressure ratio $\left(\frac{P'}{P_0'}\right)$ if the initial vorticity parameter t_0 be known.

It is convenient to express the results by comparison with a homogeneous gas expanding according to a pressure volume index n, i.e.,

$$\left(\frac{P'}{P_0'}\right) = \left(\frac{r_0'^2}{r'^2}\right)^n.$$



Pressure-Volume Characteristic for a Fermi-Dirac Gas.

Hence,

$$n = -ln\left(\frac{P'}{P_0'}\right) \Bigg/ ln\left(\frac{r'^2}{r_0'^2}\right). \quad \cdots \quad \cdots \quad \cdots \quad (30)$$

Substituting values in term of t and $f(t)$, we get

$$n = -\frac{5}{3}\frac{ln\dfrac{t-1}{t_0-1}-ln\dfrac{f(t)}{f(t_0)}}{ln\dfrac{t-1}{t_0-1}-\dfrac{2}{3}ln\dfrac{f(t)}{f(t_0)}}. \quad \cdots \quad \cdots \quad \cdots \quad (31)$$

Now $\qquad n < \frac{5}{3}$ if $\frac{d}{dt}f(t)$ is positive

and $\qquad n > \frac{5}{3}$ if $\frac{d}{dt}f(t)$ is negative.

But

$$\frac{d}{dt}f(t) = \frac{9}{4}-\frac{3}{4}\left(\frac{3}{2}\sqrt{t}-\frac{1}{2\sqrt{t}}\right) ln\,\frac{1+\dfrac{1}{\sqrt{t}}}{1-\dfrac{1}{\sqrt{t}}},$$

$$= -\frac{9}{4}\left(\frac{1}{3t}+\frac{1}{5t^2}+\frac{1}{7t^3}+\dots\right)$$

$$+\frac{3}{8}\left(\frac{1}{t}+\frac{1}{3t^2}+\frac{1}{5t^3}+\dots\right)$$

$$< 0.$$

Hence,

$$n > \frac{5}{3}.$$

Thus we see that the presence of vorticity in a non-relativistic Fermi-Dirac gas effectively increases the value of n.

The value of n has been calculated for initial vorticity parameters t_0 equal to 1·025, 1·05, 1·10 and 1·20 and is plotted in the figure. It will be observed that the amount of increase in n is chiefly dependent on the initial degree of vorticity.

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Summary

We have investigated in this note the effect of vorticity on the specific heat ratio of a non-relativistic Fermi-Dirac gas. We find that the ratio increases with vorticity.

References

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