ON A GENERATING FUNCTION IN PARTITION THEORY

by Hansraj Gupta, *F.N.I., Panjab University College, Hoshiarpur*

(*Received June 7; read August 6, 1954*)

1. In this note, we shall be concerned with the partial fractions of

$$f(x, r) = r\,!\,/(1-x)(1-x^2)(1-x^3)\ldots(1-x^r)\,;\ |x| < 1.$$

This function is of importance in the Theory of Partitions. In fact, it is well known that

$$\frac{1}{r\,!}f(x, r) = \sum_{n=0}^{\infty} p(n, r)x^n, \qquad \ldots \qquad (1)$$

where $p(n, r)$ denotes for $n > 0$, the number of partitions of n into at the most r summands and $p(0, r) = 1$.

2. Let

$$f(x, r) = \sum_{j=0}^{\infty} A_j(r)\,.\,(1-x)^{j-r}\,;$$

so that

$$y^r f(1-y, r) = \sum_{j=0}^{\infty} A_j(r)y^j, \quad 0<y<2. \qquad \ldots \qquad (2)$$

Then, since

$$f(x, r) = \frac{r}{1-x^r}f(x, r-1),$$

we must have

$$\{1-(1-y)^r\}f(1-y, r) = r\,f(1-y, r-1);$$

or

$$\frac{1}{r}\left\{\sum_{i=1}^{r}(-1)^{i-1}\binom{r}{i}y^{i-1}\right\}.\sum_{j=0}^{\infty}A_j(r)\,.\,y^j = \sum_{j=0}^{\infty}A_j(r-1)\,.\,y^j. \qquad \ldots \qquad (3)$$

Comparing the coefficients of the powers of y on the two sides of (3), we readily obtain

$$\binom{r}{1}A_j(r)-\binom{r}{2}A_{j-1}(r)+\binom{r}{3}A_{j-2}(r)-\ldots+(-1)^j\binom{r}{j+1}A_0(r) = rA_j(r-1). \qquad (4)$$

Hence

$$A_j(r) = \sum_{k=2}^{r}\frac{1}{k}\left\{\binom{k}{2}A_{j-1}(k)-\binom{k}{3}A_{j-2}(k)+\ldots+(-1)^{j+1}\binom{k}{j+1}A_0(k)\right\}, \qquad (5)$$

with $A_0(r) = 1$.

In particular,

$$A_1(r) = \sum_{k=2}^{r} \frac{1}{k}\binom{k}{2} = \sum_{k=2}^{r} \frac{1}{2}\binom{k-1}{1} = \frac{1}{2}\binom{r}{2}.$$

$$A_2(r) = \sum_{k=2}^{r} \left\{\frac{1}{2}\binom{k-1}{1} A_1(k) - \frac{1}{3}\binom{k-1}{2} A_0(k)\right\};$$

$$= \sum_{k=2}^{r} \left\{q_2(2)\binom{k-1}{3} + q_1(2)\binom{k-1}{2} + q_0(2)\binom{k-1}{1}\right\}; \text{ say.}$$

To evaluate the q's, we put in turn for k, the values 2, 3, 4.. in the two expressions on the right; thus

$$q_0(2) = \frac{1}{2}A_1(2) = \frac{1}{4};$$

$$q_1(2) = A_1(3) - \frac{1}{3}A_0(3) - 2q_0(2) = \frac{2}{3};$$

$$q_2(2) = \frac{3}{2}A_1(4) - A_0(4) - 3q_1(2) - 3q_0(2) = \frac{3}{4}.$$

Hence

$$A_2(r) = \frac{3}{4}\binom{r}{4} + \frac{2}{3}\binom{r}{3} + \frac{1}{4}\binom{r}{2}.$$

One can now easily show that

$$A_j(r) = q_{2j-2}(j)\binom{r}{2j} + q_{2j-3}(j)\binom{r}{2j-1} + \dots + q_0(j)\binom{r}{2}; \quad \dots \quad \dots \quad (6)$$

where the q's are functions of j alone.
In view of (5), we must have

$$q_{2j-2}(j)\binom{k-1}{2j-1} + q_{2j-3}(j)\binom{k-1}{2j-2} + \dots + q_0(j)\binom{k-1}{1}$$

$$= \frac{1}{2}\binom{k-1}{1} A_{j-1}(k) - \frac{1}{3}\binom{k-1}{2} A_{j-2}(k) + \dots + \frac{(-1)^{j+1}}{j+1}\binom{k-1}{j} A_0(k)$$

$$= t_k(j), \text{ say.} \quad \dots \quad \dots \quad (7)$$

Hence

$$q_0(j) = t_2(j),$$

$$q_1(j) + \binom{2}{1} q_0(j) = t_3(j),$$

$$q_2(j) + \binom{3}{2} q_1(j) + \binom{3}{1} q_0(j) = t_4(j),$$

....

$$q_i(j) + \binom{i+1}{i} q_{i-1}(j) + \binom{i+1}{i-1} q_{i-2}(j) + \dots + \binom{i+1}{1} q_0(j) = t_{i+2}(j).$$

Eliminating $q_{i-1}(j), q_{i-2}(j), \ldots, q_0(j)$ from these relations, we get

$$q_i(j) = t_{i+2}(j) - \binom{i+1}{1} t_{i+1}(j) + \binom{i+1}{2} t_i(j) - \ldots + (-1)^i \binom{i+1}{i} t_2(j). \quad \ldots \quad (8)$$

Also comparing the coefficients of k^{2j-1} on the two sides of (7), we get

$$q_{2j-2}(j) = \frac{2j-1}{2} q_{2j-4}(j-1) = \frac{(2j-1)(2j-3)\ldots(1)}{2^j}$$

$$= \frac{(2j)!}{4^j . j!}, \quad \ldots \quad (9)$$

since $\qquad q_0(1) = \frac{1}{2}$.

3. The following tables give $A_j(r)$ for values of $j \leqslant 5$. These give also values of $A_j(r)$ for $2 \leqslant r \leqslant 12$.

TABLE I

j	$A_j(r)$
0	1.
1	$\frac{1}{2}\binom{r}{2}$.
2	$\frac{3}{4}\binom{r}{4}+\frac{2}{3}\binom{r}{3}+\frac{1}{4}\binom{r}{2}$.
3	$\frac{15}{8}\binom{r}{6}+\frac{10}{3}\binom{r}{5}+\frac{9}{4}\binom{r}{4}+\frac{2}{3}\binom{r}{3}+\frac{1}{8}\binom{r}{2}$.
4	$\frac{105}{16}\binom{r}{8}+\frac{35}{2}\binom{r}{7}+\frac{2665}{144}\binom{r}{6}+\frac{49}{5}\binom{r}{5}+\frac{45}{16}\binom{r}{4}+\frac{4}{9}\binom{r}{3}+\frac{1}{16}\binom{r}{2}$.
5	$\frac{945}{32}\binom{r}{10}+105\binom{r}{9}+\frac{11095}{72}\binom{r}{8}+\frac{602}{5}\binom{r}{7}+\frac{7805}{144}\binom{r}{6}+\frac{1289}{90}\binom{r}{5}$ $+\frac{9}{4}\binom{r}{4}+\frac{2}{9}\binom{r}{3}+\frac{1}{32}\binom{r}{2}$.

TABLE II

$r \downarrow$ \ $j \rightarrow$	0	1	2	3	4	5
2	1	1/2	1/4	1/8	1/16	1/32
3	1	3/2	17/12	25/24	91/144	91/288
4	1	3	59/12	17/3	715/144	479/144
5	1	5	155/12	45/2	20831/720	20237/720
6	1	15/2	85/3	425/6	31037/240	85823/480
7	1	21/2	329/6	1127/6	112357/240	85225/96
8	1	14	581/6	875/2	344897/240	18039/5
9	1	18	319/2	1843/2	929377/240	149845/12
10	1	45/2	995/4	14335/8	1127777/120	9108953/240
11	1	55/2	1485/4	26125/8	1256431/60	24887203/240
12	1	33	6413/12	33847/6	3911809/90	9333863/36

4. Some asymptotic Results.

From (9) it readily follows that for any fixed j and a sufficiently large r,

$$A_j(r) \sim \frac{r^{(2j)}}{4^j . j\,!}, \text{ where } r^{(k)} = r(r-1)(r-2) \ldots (r-k+1). \qquad .. \quad (10)$$

Expanding (2) by the binomial theorem, we have for any fixed r and a sufficiently large n,

$$r\,!\,p(n, r) \sim \sum_{j=0}^{r-1} A_j(r) \binom{n+r-j-1}{r-j-1}. \qquad .. \quad .. \quad (11)$$

Writing $$C_{r-j-1} \text{ for } \binom{n+r-j-1}{r-j-1},$$

and $$\widehat{\exp}\left(\frac{r^2}{4x}\right) \text{ for } 1+\frac{r^{(2)}}{4\,.\,1\,!\,x}+\frac{r^{(4)}}{4^2\,.\,2\,!\,x^2}+\frac{r^{(6)}}{4^3\,.\,3\,!\,x^3}+\ldots$$

where the symbol $\frown$ indicates that the indices of the powers of r in the expansion of $\exp\left(\frac{r^2}{4x}\right)$ are to be enclosed within brackets, we can say that $r\,!\,p(n, r)$ is asymptotically equal to the term independent of x in

$$\{ C_{r-1}+C_{r-2}\,x+C_{r-3}\,x^2+\ldots \} \;\widehat{\exp}\left(\frac{r^2}{4x}\right) \qquad .. \quad .. \quad (12)$$

These results compare favourably with those given by me earlier.

References

H. Gupta (1942). On an Asymptotic Formula in Partitions. *Proc. Indian Acad. Sci.*, **16**, 101-102.

——— (1943). On the Maximum values of $p_k(n)$ and $\pi_k(n)$. *Jour. Indian Math. Soc.*, **7**, 72–75.

——— (1946). An Asymptotic Formula in Partitions. *Jour. Indian Math. Soc.*, **10**, 73–76.

——— (1942). An Inequality in Partitions. *Jour. Bombay Univ.*, 11/3, 16–18.

Issued November 17, 1954.