ON THE EXISTENCE OF HYDROGEN ATMOSPHERES IN RED GIANT STARS

by U. R. Burman, *Department of Applied Mathematics, University of Calcutta*

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1. Introduction

The study of the problem of the internal constitution of the red giant stars has received considerable attention in recent years. These stars, found to be scattered rather irregularly in a belt which branches towards the right off the upper region of the main sequence in the Hertzsprung-Russell diagram, are characterised by high luminosities in spite of their comparatively low surface temperatures. Since low surface temperature implies a rather small intensity of light emitted per unit surface, the high luminosities can be understood only on the hypothesis of extremely large dimensions. It was pointed out by Öpik that an increase in the radius of a normal star might be obtained by assuming that the star has a central convective core with a higher value of the molecular weight than the surrounding radiative envelope. Hoyle and Lyttleton (1942), however, suggest that a large extension in the radius is possible only if this non-uniformity in the composition of the stellar material takes place somewhere in the radiative region of the star. These authors also remark that the desired non-uniformity in the atmospheric composition may be brought about by the accretion of hydrogen from interstellar space to the surface of the star. It is convenient for purposes of analysis to represent this non-uniformity in the composition by a sharp discontinuity of the molecular weight μ, though in an actual case there will probably be a small transition zone of continuously varying μ. If however the accretion of hydrogen is solely responsible for the non-uniformity in composition, as is contemplated here, the discontinuity in μ is indeed strictly sharp.

Hoyle and Lyttleton (1949), as well as Li Hen and Schwarzschild (1949) constructed a number of stellar models under the assumption that the model consists of *three* parts: a central convective core, an intermediate radiative region with the same chemical composition as that of the core, and an outer radiative envelope composed purely of hydrogen and hence differing in composition from the rest of the star. The inhomogeneity in chemical composition thus takes place at the interface between the intermediate zone and the hydrogen envelope. Some of these models also yield quite large extensions of radii, as demanded by the observations on red giants. None of these models, however, have been tested by reference to an actual red giant whose mass (M), radius (R) and luminosity (L) are known from observations; in other words, the models have not been used to determine the composition, the central density and temperature of the star from a knowledge of its L, M and R. Unless a model conforms to this requirement, it must be regarded as a tentative one. There is also another point in the construction of these models which deserves some careful consideration. The opacity of the stellar material has been assumed to be given by Kramers' formula duly corrected for the guillotine factor, in both the intermediate zone and the hydrogen envelope. This assumption is not, however, free from objection in view of the fact that the opacity of a pure hydrogen atmosphere is not given by Kramers' law in the usual form.

The purpose of the present paper is to examine whether a three phase stellar configuration of the type discussed above can be built up with an opacity in the outer region appropriate to highly ionised hydrogen, Kramers' formula for opacity being, however, retained to govern the structure of the intermediate radiative zone, the central core as usual being in convective equilibrium. The model should also be able to furnish complete information regarding the internal constitution of the star from a knowledge of its three observable parameters, L, M and R. Further, the possibility of constructing a two-phase configuration, namely, a configuration with a central convective core surrounded by an envelope of pure hydrogen in radiative equilibrium, has also been considered. It is found that a configuration consistent with all the conditions of thermodynamic and mechanical equilibrium, and with an energy output * of the prescribed amount, does not exist under the assumptions involved. This conclusion leads one to believe that the outer layers of a giant star are probably not composed of hydrogen alone. An admixture in suitable proportions of other elements, with hydrogen in the outer layers of the star, may be regarded as an alternative hypothesis.

2. The Basic Assumptions and the Structure of the Intermediate Zone

As outlined above, the stellar model has been assumed to consist of three zones as follows:—

(i) a central convective core within which almost the entire energy generation takes place,
(ii) an intermediate zone in radiative equilibrium having the same chemical composition as that of the core, and
(iii) an outer envelope of pure hydrogen also in radiative equilibrium.

The model would thus have a single discontinuity of composition occurring somewhere in the radiative region. In accordance with the usual convention, we shall regard the perfect gas law as valid throughout the configuration and shall ignore the influence of radiation pressure. Kramers' law for photoelectric opacity as modified by Li Hen and Schwarzschild (1949) to account for the variation of the guillotine factor, will be adopted for the intermediate radiative zone, while an appropriate opacity law for highly ionised hydrogen will govern the structure of the outer envelope.

The equilibrium equations in the intermediate zone embodying the above assumptions can be written down in the usual notations as

$$\frac{dP}{dr} = -G\frac{M(r)}{r^2}\rho, \qquad (1)$$

$$\frac{dM(r)}{dr} = 4\pi r^2\rho, \qquad (2)$$

$$\frac{d}{dr}(\tfrac{1}{3}aT^4) = -\frac{\kappa\rho}{c}\frac{L}{4\pi r^2}. \qquad (3)$$

The opacity coefficient, according to Li Hen and Schwarzschild (1949), is given by

$$\kappa = \kappa_0\rho^{0.75}T^{-3.5}, \text{ with } \kappa_0 = 10^{25}(1+X)(1-X-Y), \qquad (4)$$

where X, Y denote respectively the hydrogen and helium contents in the material.

* The carbon-nitrogen cycle and the proton-proton reaction have both been considered as energy generating mechanisms in this connection.

In terms of the dimensionless variables p, t, q, x (Schwarzschild, 1946) defined by

$$P = p\frac{GM^2}{4\pi R^4},\ T = t\frac{\mu H}{k}\frac{GM}{R},\ M(r) = qM,\ r = xR, \quad \ldots \quad (5)$$

the equations (1)–(3) become

$$\frac{dp}{dx} = -\frac{p}{t}\frac{q}{x^2}, \quad \ldots \quad (6)$$

$$\frac{dq}{dx} = \frac{p}{t}x^2, \quad \ldots \quad (7)$$

$$\frac{dt}{dx} = -\frac{C}{x^2}\frac{p^{1\cdot75}}{t^{8\cdot25}}, \quad \ldots \quad (8)$$

where

$$C = \frac{3\kappa_0}{4ac}\left(\frac{k}{\mu HG}\right)^{7\cdot5}\left(\frac{1}{4\pi}\right)^{2\cdot75}\frac{LR^{1\cdot25}}{M^{5\cdot75}}. \quad \ldots \quad (9)$$

Introducing the following substitutions in equations (4)–(6),

$$\lambda = \log\frac{p}{p_0},\ \psi = \log\frac{q}{q_0},\ \tau = \log\frac{t}{t_0},\ y = \log\frac{x}{x_0}, \quad \ldots \quad (10)$$

and assuming that the *four* constants with zero suffix can be chosen so as to satisfy the three conditions

$$\frac{q_0}{t_0x_0} = 1,\ \frac{p_0x_0^3}{t_0q_0} = 1,\ \frac{Cp_0^{1\cdot75}}{t_0^{9\cdot25}x_0} = 1, \quad \ldots \quad (11)$$

Li Hen and Schwarzschild (1949) re-write the equations in the forms

$$\log\left(-\frac{d\lambda}{dy}\right) = \psi-\tau-y, \quad \ldots \quad (12)$$

$$\log\left(\frac{d\psi}{dy}\right) = \lambda-\tau-\psi+3y, \quad \ldots \quad (13)$$

$$\log\left(-\frac{d\tau}{dy}\right) = 1\cdot75\,\lambda-9\cdot25\,\tau-y. \quad \ldots \quad (14)$$

These equations are to be integrated outwards with starting values of the variables appropriate to the conditions of fit obtaining at the interface between the radiative zone and the convective core which is an Emden polytrope of index $n = 1\cdot5$. Since there is no discontinuity of composition across this interface, the homology invariants

$$U = \frac{d\log M(r)}{d\log r},\ V = -\frac{d\log P}{d\log r},\ n+1 = \frac{d\log P}{d\log T} \quad \ldots \quad (15)$$

must be continuous across it. Using these conditions Li Hen and Schwarzschild (1949) have integrated the equations (12)–(14) and given the solutions in tabular forms. These solutions which depend on one parameter ξ_1 (ξ is the independent Emden variable, i.e., the radial distance for the polytrope $n = 1\cdot5$) defining the radius of the central convective core and chosen arbitrarily, determine uniquely the structure of the intermediate zone.

3. The Structure of the Hydrogen Envelope

The structure of the hydrogen envelope is governed by equations (1), (2) and (3), except that in equation (3) the opacity coefficient κ has the value appropriate to highly ionised hydrogen, viz.

$$\kappa = BPT^{-\frac{11}{2}} e^{\frac{\chi_1}{kT}}, \qquad \text{(16)}$$

where B is a numerical constant and χ_1 the ionisation potential of hydrogen. This formula for the hydrogen opacity has been obtained by Wasiutinsky (1946) and has been used by him to find an approximate * pressure-temperature relation in the hydrogen envelope in the form

$$P = \frac{2}{\sqrt{19\alpha}} T^{\frac{19}{4}} e^{-\frac{21}{38}\frac{\chi_1}{kT}}, \qquad \text{(17)}$$

where

$$\alpha = \frac{3BL}{16\pi acGM}. \qquad \text{(18)}$$

We shall now use equation (17) to calculate the temperature distribution in this region.

Inserting the value of κ from equation (16) in the radiative equilibrium equation (3), and using the perfect gas law

$$P = \frac{k}{\mu H}\rho T,$$

one obtains

$$\frac{dT}{dr} = -\frac{3BL}{16\pi ac}\frac{\mu H}{k} T^{-\frac{19}{2}} e^{\frac{\chi_1}{kT}} \frac{P^2}{r^2},$$

which with the help of equations (17) and (18) can be written as

$$\frac{dT}{dr} = -\frac{4}{19} GM \frac{\mu H}{k} e^{-\frac{2}{19}\frac{\chi_1}{kT}} \cdot \frac{1}{r^2}.$$

An integration of this equation, taking $\mu = \frac{1}{2}$ leads to

$$\frac{2}{19} GM \frac{H}{k}\left(\frac{1}{r} - \frac{1}{R}\right) = \int_{T_s}^{T} e^{\frac{2}{19}\frac{\chi_1}{kT}}\, dT$$

$$= \left[Te^{\frac{2}{19}\frac{\chi_1}{kT}} - \frac{2}{19}\frac{\chi_1}{k}\left\{ \log\left(\frac{2}{19}\frac{\chi_1}{kT}\right) + \frac{2}{19}\frac{\chi_1}{kT}\frac{1}{1!} + \left(\frac{2}{19}\frac{\chi_1}{kT}\right)^2 \frac{1}{2.2!} + \left(\frac{2}{19}\frac{\chi_1}{kT}\right)^3 \frac{1}{3.3!} + \ldots \right\} \right]_{T_s}^{T}, \qquad \text{(19)}$$

where T_s refers to the surface temperature of the star assumed to be non-zero to avoid the divergence in the integral.

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* The variation of mass in the region under consideration has been ignored in this derivation.

The mass distribution inside the hydrogen region can now be obtained as

$$M(r) = M + \int_R^r 4\pi r^2 \rho \, dr$$

$$= M + \frac{4\pi}{\sqrt{19\alpha}} \frac{H}{k} \int_R^r T^{\frac{15}{4}} e^{-\frac{21}{38}\frac{\chi_1}{kT}} r^2 \, dr. \quad \ldots \quad (20)$$

The integral in equation (20) is to be evaluated numerically by using the T-r table that may be constructed from equation (19).

4. The Equations of Fit

At the interface (where the discontinuity in composition occurs) between the intermediate radiative zone and the hydrogen envelope, the usual conditions for the continuity of r, $M(r)$, P and T expressed in terms of the dimensionless variables in equation (10) lead to the following relations,

$$y_2 = \log \frac{r_2}{Rx_0}, \quad \ldots \quad (21)$$

$$\psi_2 = \log \frac{M(r_2)}{q_0 M}, \quad \ldots \quad (22)$$

$$\lambda_2 = \log \left\{ P_2 \bigg/ \left(p_0 \frac{GM^2}{4\pi R^4} \right) \right\} = \log \frac{p_2}{p_0}, \quad \ldots \quad (23)$$

$$\tau_2 = \log \left\{ T_2 \bigg/ \left(t_0 \frac{GM}{R} \frac{\mu H}{k} \right) \right\}, \quad \ldots \quad (24)$$

where the suffix 2 refers to the interface under consideration and μ denotes the constant molecular weight in the entire region interior to this interface. The four constants x_0, q_0, p_0, t_0 are connected by equations (11). There is the further condition of the continuity of $L(r)$ across the interface, which may be obtained thus—

The equations of hydrostatic and radiative equilibrium in the hydrogen envelope give

$$\frac{d(\frac{1}{3}aT^4)}{dP} = \frac{\kappa L(r)}{4\pi c G M(r)},$$

which with the help of equation (17) may be written in the form

$$\frac{L(r)}{M(r)} = \frac{1}{n+1} \frac{16\pi acG}{3B} \frac{T^{\frac{19}{2}}}{P^2} e^{-\frac{\chi_1}{kT}}$$

$$= \frac{16\pi acG}{3B} \frac{T^{\frac{19}{2}}}{P^2} e^{-\frac{\chi_1}{kT}} \left(\frac{19}{4} + \frac{21}{38} \frac{\chi_1}{kT} \right)^{-1}. \quad \ldots \quad (25)$$

In the intermediate radiative zone, one similarly obtains

$$n+1 = \left(\frac{d \log P}{d \log T} \right)_{\rm rad} = \frac{16\pi acG}{3\kappa_0} \frac{T^{7\cdot5}}{P\rho^{0\cdot75}} \frac{M(r)}{L(r)}$$

$$= \frac{16\pi acG}{3\kappa_0} \frac{T}{P}^{7\cdot5} \left(\frac{k}{\mu H} \right)^{0\cdot75} \left(\frac{T}{P} \right)^{0\cdot75} \frac{M(r)}{L(r)}. \quad \ldots \quad (26)$$

Equality of $\frac{L(r)}{M(r)}$ from equations (25) and (26) therefore gives the desired condition at the interface as

$$\left(\frac{k}{H}\right)^{0\cdot75} \frac{1}{\mu^{0\cdot75}\kappa_0(n_i+1)} = e^{-\frac{\chi_1}{kT_2}} \left\{ B\left(\frac{19}{4}+\frac{21}{38}\frac{\chi_1}{kT_2}\right)\right\}^{-1} \frac{T_2^{1\cdot25}}{P_2^{0\cdot25}}, \quad .. \quad (27)$$

where the suffix i denotes the value on the inner side of the interface.

5. Construction of the Model

We have seen how with an assumed set of values for L, M and R, the structure of the hydrogen envelope can be completely determined. The compilation of the complete model may then be proceeded with with the help of the homology invariants U and V. Starting with the definitions of U, V and using equations (1) and (2), one easily obtains

$$\frac{U}{V} = \frac{4\pi}{G} \frac{r^4 P}{\{M(r)\}^2}, \quad .. \quad .. \quad .. \quad .. \quad (28)$$

which shows that the ratio $\frac{U}{V}$ is continuous across the interface between the hydrogen envelope and the intermediate radiative zone, i.e. across the surface of discontinuity of μ. Since the quantities occurring on the right of equation (28) can all be calculated at any point inside the hydrogen region, the ratio $\frac{U}{V}$ will therefore be known at every point of it. Further, this ratio can also be calculated at any point of the intermediate zone, corresponding to each of the tabulated solutions provided by Li Hen and Schwarzschild (1949). Starting therefore from the surface of the star, we follow the hydrogen envelope solution, terminate it at an arbitrary point r_2, and pass on to any chosen solution (from the tables of Li Hen and Schwarzschild) of the intermediate zone characterised by its parameter ξ_1 and satisfying the continuity condition for $\frac{U}{V}$ at the point r_2 where we break off the envelope solution. Along any solution of the intermediate zone, the ratio $\frac{U}{V}$ decreases steadily outwards. These solutions have the further property, as will be evident from the integrations of Li Hen and Schwarzschild (1949), that some of them ($\xi_1 < 1\cdot12009$) show a steadily increasing n outwards and ultimately behave like isothermals ($n \to \infty$), while for others ($\xi_1 > 1\cdot12009$) n increases steadily at first, attains a maximum and then gradually diminishes. These latter solutions will cease to be applicable for a fit with the hydrogen envelope after n drops below $1\cdot5$. It appears therefore quite possible that within the range of applicability of these solutions, they may not be fitted on, as far as the continuity of the ratio $\frac{U}{V}$ is concerned, to the envelope solution at an arbitrary point. When, however, the fit is possible, at the junction of the two solutions, the variables y_2, ψ_2, λ_2, τ_2 for the intermediate zone will all be known and equations (21)–(24) will then fix up the values of the quantities x_0, q_0, p_0 and μt_0 appropriate to the fit. It should, however, be observed that the quantities x_0, q_0, p_0 thus determined must conform to the restriction

$$q_0^2 = p_0 x_0^4, \quad .. \quad .. \quad .. \quad .. \quad .. \quad (29)$$

imposed on them by the first two of equations (11). A suitable choice of the position of the surface of discontinuity of μ will enable one to secure this adjustment without much difficulty. The condition

$$\frac{q_0}{t_0 x_0} = 1$$

in equation (11) then determines the value of t_0 which in conjunction with the already known value of μt_0 would fix up the value of μ, the molecular weight of the material in the region interior to the hydrogen envelope. The composition parameters (viz. the hydrogen and helium contents) as well as the central density and temperature are now to be found for the complete model. This is done as follows:—

The condition (Equation (11))

$$\frac{Cp_0^{1\cdot 75}}{t_0^{9\cdot 25} x_0} = 1$$

determines C. Writing

$$\mu = \frac{2}{1+3X+0\cdot 5Y} = f_1(X, Y), \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

where X, Y denote respectively the hydrogen and helium contents, equation (9) may be put in the form

$$C = f_2(X, Y, T_c, \rho_c, \xi_1). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

The energy-output equation will furnish another relation of the type

$$L = f_3(X, Y, T_c, \rho_c, \xi_1), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

T_c, ρ_c being the central temperature and density of the model sought. Finally we have equation (27) which we rewrite in the equivalent form

$$0 = f_4(X, Y, \xi_1, r_2). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (33)$$

The four equations (30)–(33) therefore fix the values of the four quantities X, Y, T_c, ρ_c and a set of physically admissible values, for these would complete the construction of the desired model.

It will be observed that the procedure adopted here involves two arbitrary choices, viz. the position (r_2) of the surface of discontinuity of composition and the size (ξ_1) of the central convective core.

6. The Energy Generation Law

While it may be generally admitted that the carbon-nitrogen cycle of Bethe is capable of explaining the energy production in the main sequence stars, the corresponding question for the red giants appears to need further investigation. We shall here examine the rôles of both the C—N cycle and the Proton-Proton reaction as mechanisms of energy liberation in the red giants. The luminosities according to the two processes are given by (Epstein (1950))

$$L_C = \frac{2\cdot 84 \times 10^7}{0\cdot 0585} \cdot \frac{1}{\mu^{\frac{3}{2}}} X\alpha_{14}\rho_c^{\frac{1}{2}} T_c^{\frac{43}{2}} \int_0^{\xi_1} \theta^{23}\xi^2 \, d\xi \quad \ldots \quad \ldots \quad (34)$$

and

$$L_P = \frac{6{\cdot}23\times 10^{25}}{0{\cdot}3396}\cdot\frac{1}{\mu^{\frac{3}{2}}}X^2\rho_c^{\frac{1}{3}}T_c^{-\frac{11}{2}}\int_0^{\xi_1}\theta^7\xi^2\,d\xi, \qquad .. \qquad .. \quad (35)$$

where α_{14} is the relative abundance of N^{14} having the value 0·01, θ the dependent variable in the Emden polytrope of index 1·5, and T_c is expressed in millions of degrees. The integrals in equations (34), (35) are to be evaluated numerically and should become sensibly constant within the convective core. The integral in equation (34) does indeed attain a nearly constant value at about $\xi = 1{\cdot}20$, while that in equation (35) does not become constant till almost the boundary of the polytrope is reached. In other words the energy generating core in the P-P reaction will be unusually large and this will have the effect of violating the equations of fit which must necessarily hold at the junction of the intermediate radiative zone and the hydrogen envelope. In fact we shall see that no model of the desired type can exist with the core radius (ξ_1) exceeding 1·20. It therefore follows that the P-P reaction alone is incapable of accounting for the energy generation in a model of the type considered here. Under the present assumptions therefore the C—N cycle must be looked upon as the probable mechanism for the energy production in the red giants.

7. Numerical Results

It is now necessary to carry out numerical calculations according to the scheme outlined in the previous sections in respect of some observed red giants. But, unfortunately the observational data regarding masses of these stars appear to be somewhat uncertain and of a very limited extent. However the best available data seem to be provided by the red components of the two binaries, Capella and Zeta Auriga. The direct observational value of four solar masses for the red component of Capella has recently been called into question by Struve (1951), who ascribes to it a mass value of about 2·7 in solar units. This new value also in its turn is not free from uncertainty.

For the red component of Zeta Auriga, a mass value around 15 solar masses is furnished by observational data. This observational result is based on the determination of the mass function and the mass ratio, which latter appears to be quite unreliable on account of a very limited range of observation. In view of these uncertainties, we have adopted the earlier observed values for the mass, radius and luminosity of the red component of Capella for the purpose of calculations in the present paper. Calculations for the red component of Zeta Auriga have not been undertaken on the same ground of uncertainty.

The observed values (in solar units) for the red component of Capella are

$$\log L = 2{\cdot}08, \quad \log M = 0{\cdot}64, \quad \log R = 1{\cdot}02.$$

With these values the surface temperature T_s' calculated from the formula

$$\frac{L}{4\pi R^2} = \frac{1}{4}acT_s^4,$$

comes out to be $T_s = 4{\cdot}79\times 10^{-3}$ expressed in millions of degrees. The temperature distribution inside the hydrogen envelope is now given by equation (19) and is shown in Table I.

With the temperature distribution in Table I, the integral in equation (20) is evaluated numerically, and the mass distribution obtained as in Table II. The values

TABLE I

Temperature distribution inside the hydrogen envelope

$\frac{\chi_1}{k} = 1{\cdot}571 \times 10^6$

$T.10^{-6}$	$\frac{r}{R}$	$\frac{r}{R}$	$T.10^{-6}$
$4{\cdot}79 \times 10^{-3}$	1·0	1·00	$4{\cdot}79 \times 10^{-3}$
$1{\cdot}0 \times 10^{-2}$	0·912	0·90	$1{\cdot}25 \times 10^{-2}$
0·1	0·755	0·80	$6{\cdot}25 \times 10^{-2}$
0·2	0·667	0·75	$10{\cdot}25 \times 10^{-2}$
0·3	0·600	0·70	$14{\cdot}76 \times 10^{-2}$
0.4	0·546	0·65	$21{\cdot}24 \times 10^{-2}$
0·5	0·502	0·60	0·30
0·6	0·464	0·55	0·40
0·7	0·432	0·50	0·50
0·8	0·404	0·47	0·56
0·9	0·380	0·44	0·65
1·0	0·358	0·41	0·75
1·2	0·322	0·38	0·90
1·6	0·267	0·35	1·05
2·0	0·229	0·32	1·20
		0·30	1·34
		0·28	1·49
		0·26	1·67
		0·24	1·87
		0·22	2·11
		0·20	2·40

of μ and T_c are then computed in some sample cases corresponding to arbitrary choices inside the model of the position at which the composition changes, and the size of the central convective core. These values are listed in Table III.

TABLE II

Distribution of mass inside the hydrogen envelope

$\frac{r}{R}$	$\frac{M(r)}{M}$
1·00	1·00
0·70	1·00
0·50	0·9694
0·44	0·9254
0·41	0·8906
0·38	0·8206
0·35	0·7096
0·32	0·5587
0·30	0·3580
0·28	0·2330
0·26	0·0346

TABLE III

The values of μ and T_c in some sample cases corresponding to arbitrary choices of r_2 and ξ_1

ξ_1	$\frac{r_2}{R}$	$\frac{U}{V}$	$\frac{r_1}{R}=x_0$ (radius of convective core)	μ	$T_c.10^{-6}$
1·1189	0·32	0·7865	0·2043	0·288	1·90
	0·35	0·3663	0·1694	0·339	2·09
	0·38	0·1804	0·1445	0·356	2·23
	0·41	0·0857	0·1142	0·296	2·08
	0·44	..	..	..	..
1·1204	0·32	0·7865	0·2043	0·290	1·90
	0·35	0·3663	0·1695	0·344	2·11
	0·38	0·1804	0·1461	0·375	2·36
	0·41	0·0857	..	..	..
1·1213	0·32	0·7865	0·2044	0·292	1·91
	0·35	0·3663	0·1696	0·354	2·17
	0·38	0·1804	0·1495	0·396	2·46
	0·41	0·0857	..	..	..
1·1250	0·32	0·7865	0·2046	0·294	1·92
	0·35	0·3663	0·1714	0·360	2·21
	0·38	0·1804	..	..	..
1·1400	0·32	0·7865	0·2075	0·302	1·97
	0·35	0·3663	..	..	..
1·2000	0·32	0·7865	..	..	..

8. Discussion of the Results

The missing figures in Table III imply that the equations of fit admit of no solution in the corresponding cases, because along the relevant intermediate zone solution defined by ξ_1, the ratio $\frac{U}{V}$ does not attain the requisite value before n drops below 1·5. This feature will be common to all intermediate zone solutions defined by $\xi_1 > 1{\cdot}12009$, while for solutions with $\xi_1 < 1{\cdot}12009$, the fit with an hydrogen envelope as far as the continuity of $\frac{U}{V}$ is concerned, can be obtained at any place inside the model. It is also easy to see that for any intermediate solution of the former class ($\xi_1 > 1{\cdot}12009$), if the continuity condition for $\frac{U}{V}$ becomes invalidated at a particular point r_2 of the hydrogen atmosphere, it will become so at all points further outwards. For this family of intermediate solutions therefore it has not been possible to extend the calculations beyond (outwards) $\frac{r_2}{R} = 0{\cdot}38$ roughly. For the latter class of intermediate solutions ($\xi_1 < 1{\cdot}12009$), however, the fit equations can be solved even beyond

$\frac{r_2}{R} = 0{\cdot}38$, but the computed values of μ and T_c will gradually diminish and render the fitted solutions physically inadmissible. It has also been verified that for both families of the intermediate solutions considered here, the initial value of the ratio $\frac{U}{V}$, viz., the value on the surface of the convective core, is less than the value of the corresponding quantity at any point given by $\frac{r_2}{R} < 0{\cdot}32$ (approximately) inside the hydrogen envelope. This remark will not, however, be applicable to intermediate solutions characterised by ξ_1 sufficiently smaller than 1·1189, but such small values of ξ_1 will not be compatible with the hypothesis that there shall be no appreciable energy generation outside the convective core. This shows that a fit of the hydrogen envelope with the intermediate region cannot be obtained at a distance smaller than 32 per cent of the star's radius. A solution of the fit equations, if any, should thus be sought for in the range $0{\cdot}32 \leqslant \frac{r_2}{R} \leqslant 0{\cdot}38$. It is here necessary to emphasise that the different solutions of the equations of fit constitute a two-parametric family, as will be evident from Table III which shows that corresponding to a given position of the surface of discontinuity of composition, there are more than one values for the radius of the central convective core.

9. Impossibility of the Three-phase Configuration

The results in Table III have been presented graphically in Fig. 1, where μ has

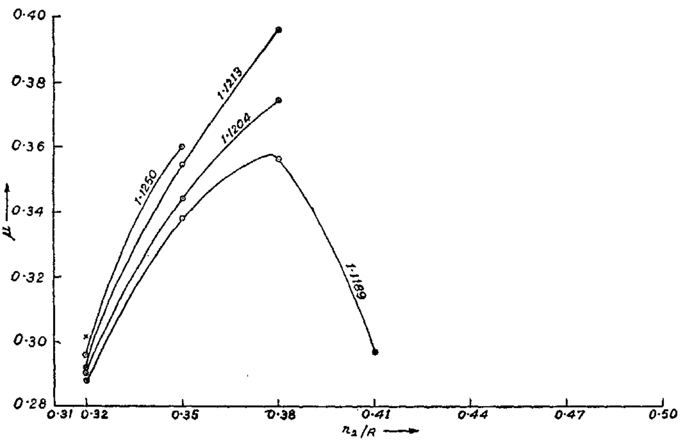

Fig. 1. The numbers alongside the curves refer to values of the radii of the convective cores (ξ_1) corresponding to the computed models. The point marked (×) refers to $\xi_1 = 1{\cdot}1400$.

been plotted against $\frac{r_2}{R}$. It will be observed that the computed values of μ corresponding to either a constant value of ξ_1 or a constant value of $\frac{r_2}{R}$ are all less than 0·5. These values of μ are clearly physically inadmissible because for any medium different from pure hydrogen the average molecular weight cannot drop below 0·5. Any solution of the stellar equations that envisages a model of this type must therefore be discarded as being contrary to physical reality. Moreover, the central temperatures of these models are found to be so low (2-3 million degrees) that thermonuclear reactions involving energy releases can hardly occur under such conditions. Even if it is admitted that the proton cycle of energy generation is operative at these low temperatures, it will not be possible to account for the high luminosity of the star unless a value of the central density in the neighbourhood of 10^{12} gm./cm.^{3}* is permitted for the model. Such high values of the central density would inevitably imply a breakdown of the perfect gas equation of state which is the very basis of the present investigation.

There is no doubt that particular values for L, M and R appropriate to an actual red giant star have been used in these calculations, but it has been verified that the calculations are not quite sensitive to these L, M, R values. Furthermore, the results do not also depend on the form of the energy generation law, viz. the C-N cycle or the P-P reaction. In fact the opacity law for pure hydrogen which prevents a rapid rise of temperature inwards from the surface of the star is responsible for these extraordinarily low central temperatures for the computed models. A three-phase configuration with an outer atmosphere of pure hydrogen cannot therefore be regarded as a possible model for the red giant stars.

10. The Two-phase Configuration

It is now necessary to examine whether a two-phase configuration consisting of a central convective core (a 3/2-polytrope) surrounded by a hydrogen envelope in radiative equilibrium can serve as a possible model for a red giant. The discontinuity in composition occurs on the surface of the convective core and the conditions to be satisfied here are

$$\left(\frac{U}{V}\right)_e = \left(\frac{U}{V}\right)_i' \qquad (36)$$

$$T_e = T_c\theta_i \qquad (37)$$

$$r_e = \left(\frac{5k}{8\pi\mu GH}\frac{T_c}{\rho_c}\right)^{\frac{1}{2}} \xi_i \qquad (38)$$

$$M(r_e) = 4\pi\rho_c\left(\frac{5k}{8\pi\mu GH}\frac{T_c}{\rho_c}\right)^{\frac{3}{2}}\left(-\xi^2\frac{d\theta}{d\xi}\right)_i, \qquad (39)$$

where the indices e and i refer respectively to the envelope and core side values on the common boundary.

The energy output equation will furnish a relation of the form

$$f(X, Y, T_c, \rho_c, \xi_i) = 0 \qquad (40)$$

and also there will be another relation analogous to equation (33) and of the form

$$F(X, Y, \xi_i) = 0. \qquad (41)$$

Equations (36)–(40) would enable one to determine the five quantities ξ_i, X, Y, T_c and ρ_c and equation (41) would serve as a check to test the correctness of the fit.

Calculations on exactly similar lines as in the three-phase configuration have been made for the red component of Capella using the P-P reaction as the energy generating mechanism and the results given in Table IV.

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* An actual application of formula (35) under assumed conditions $T_c \sim 4$ millions. $X \sim 1$, $\mu \sim 0{\cdot}5$ leads to this result.

TABLE IV

Approximate values of the central temperatures and densities for the two-phase configurations

$\frac{r_s}{R}$	ξ_i (convective core radius in Emden variable)	$T_c \cdot 10^{-6}$	ρ_c (gm./cm.^{3})
0·70	3·60	13·0	$1{\cdot}5 \times 10^{7}$
0·60	3·50	6·8	$2{\cdot}5 \times 10^{10}$
0·50	3·20	5·0	$8{\cdot}0 \times 10^{11}$
0·40	2·90	4·0	$9{\cdot}1 \times 10^{12}$
0·30	1·11	1·6	$1{\cdot}5 \times 10^{17}$

It will appear from Table II that the mass of the hydrogen envelope up to a depth of about 30 per cent of the star's radius may be regarded as negligibly small, and under this circumstance a two-phase configuration with a convective core extending beyond 70 per cent of the star's radius would hardly have any physical significance. Calculations in Table IV have not therefore been extended beyond this range. The orders of the computed central densities and temperatures definitely indicate a breakdown of the perfect gas law, so that a two-phase configuration in conformity with our basic assumptions cannot also serve as a red giant model. The problem, however, of the existence of a red giant model (with no convective core) in radiative equilibrium at the centre and having a hydrogen atmosphere also in radiative equilibrium has not been attempted here for want of a lack of more detailed knowledge regarding the integration of the radiative equations from the centre outwards.

Lastly there remains the question whether it is possible to construct a red giant model with pure hydrogen in radiative equilibrium throughout. The decisive answer to this question requires numerical integration of the stellar equations for the appropriate hydrogen opacity which is not available at the present moment, but from the temperature distribution inside the hydrogen envelope we have considered, one may reasonably conjecture that the central temperatures of the integrated models would be too low to account for the high luminosities of the red giants. Moreover, there is also the difficulty of thermonuclear energy generation in a pure hydrogen star.

Therefore the conclusion may be drawn that red giant stars cannot possess atmospheres of pure hydrogen.

SUMMARY

This paper attempts to answer the question whether a three-phase configuration consisting of a central convective core, an intermediate radiative zone (having the same chemical composition as that of the core) and an outer envelope of *pure* hydrogen also in radiative equilibrium, can serve as a suitable model for a red giant star, whose mass, radius and luminosity are known from observations. An opacity formula appropriate to highly ionised hydrogen has been used for the outer envelope, while the usual Kramers' law of opacity has been assumed for the intermediate region. The central temperatures for the computed models satisfying all these conditions are found to be extremely low (about 2-3 million degrees), so as to be totally inadequate to account for the high luminosities of these stars by any of the known mechanisms of thermonuclear reactions. A two-phase configuration consisting of a central convective core surrounded by a radiative hydrogen envelope has also been considered in this connection. It is found that here also the equations furnish no physically admissible solutions.

It is, however, known that three-phase configurations can be built up with central temperatures appropriate for thermonuclear energy generation, when Kramers' law of opacity is assumed to hold throughout the model outside the central convective core. The different

behaviour obtained in the present case is clearly to be attributed to the opacity governing the structure of the hydrogen envelope.

It is thus reasonable to conclude that under the conditions envisaged here an atmosphere of pure hydrogen is not compatible with the structure of a red giant star.

References

Epstein, I. (1950). A note on energy generation. *Ap. J.*, **112**, 207.
Hen, Li, and Schwarzschild, M. (1949). Red giant models with chemical inhomogeneities. *M.N.*, **109**, 631.
Hoyle, F. and Lyttleton, R. A. (1942). On the nature of red giant stars. *M.N.*, **102**, 218.
Hoyle, F. and Lyttleton, R. A. (1949). The structure of stars of non-uniform composition. *M.N.*, **109**, 614.
Schwarzschild, M. (1946). On the helium content of the sun. *Ap. J.*, **104**, 203.
Struve, O. (1951). The spectrum of Capella. *Proc. Nat. Acad. Sci.*, **37**, 327.
Wasiutinsky, J. (1946). The structure of the hydrogen envelope in radiative equilibrium. *Astrophysica Norvegica*, **4**, 443.

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