THE 'TRANSLATIVITY' PROBLEM FOR QUASI-HAUSDORFF METHODS OF SUMMABILITY

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Abstract

The paper deals with the problem of translativity of quasi-Hausdorff methods defining series-to-series transformations, and in Theorem I, a sufficient condition is proved for the δ_0-matrix (H^*, μ_n) to be translative for the class of series with bounded partial sums. The proof of the theorem indicates the relaxation of the above condition of boundedness of the partial sums in particular cases, where the matrix of transformation satisfies certain restrictions. These are indicated in the remarks following Theorem I. The results on the translativity of the method $A(p)$ of Taylor series continuation are deduced.

1. Introduction

Consider any transformation of the series Σu_n into the series Σv_n, defined by

$$v_n = \sum_{k=0}^{\infty} h_{nk} u_k. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

The transformation is said to be *conservative* and the matrix $H \equiv (h_{nk})$ a δ-*matrix* if Σv_n converges to a limit whenever Σu_n converges to a limit, the two limits not necessarily being the same. If, in addition, the limits are also the same, the transformation is said to be *regular* and the matrix an α-*matrix*. A special δ-matrix and a special α-matrix, in each of which $\lim_{k \to \infty} h_{nk} = 0$, are called a δ_0-*matrix* and an α_0-*matrix* respectively.

In the case of the more familiar transformation

$$t_n = \sum_{k=0}^{\infty} a_{nk} s_k, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

of $\{s_n\}$ to $\{t_n\}$ by the matrix $A \equiv (a_{nk})$, the terms '*conservative*' and '*regular*' are defined as before; and A is called a K-*matrix* or T-*matrix* according as it is conservative or regular. If the transformation is conservative and such that $\lim t_n = \rho \,.\, \lim s_n$, it is said to be *multiplicative* ρ.

The matrix $\lambda^* \equiv (\lambda^*_{nk})$ defined by

$$\lambda^*_{nk} = \binom{k}{n} \Delta^{k-n} \mu_n, \; (k > n) \;\text{ and }\; \lambda^*_{nk} = 0 \; (k < n) \quad \ldots \quad \ldots \quad (3)$$

is called a *quasi-Hausdorff matrix* and denoted by $\lambda^* \equiv (H^*, \mu_n)$.

We say that $\{\mu_n\}$ is a *moment constant* (generated by the function $\chi(t)$) if there exists a function $\chi(t)$ of bounded variation in (0, 1) such that

$$\mu_n = \int_0^1 t^n d\chi(t), \; n = 0, 1, 2, \ldots\ldots, \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

and (as we may assume without loss of generality) $\chi(0) = 0$.

If further $\chi(+0) = \chi(0) = 0$ then $\{\mu_n\}$ is called a *semi-regular moment constant*. If furthermore $\chi(1) = 1$ holds (so that $\mu_0 = 1$) then $\{\mu_n\}$ is called a *regular moment constant*.

Corresponding to any moment sequence $\{\mu_n\}$ there exists a *moment function* $T(z)$, regular for $\mathbf{R}z > 0$ and continuous and bounded for $\mathbf{R}z \geqslant 0$, defined by

$$T(z) = \int_0^1 t^z d\chi(t).$$

The function $T(z) \to \chi(1)-\chi(1-0)$ as $\mathbf{R}z \to \infty$, uniformly in $\mathbf{I}z$; and the constant $\chi(1)-\chi(1-0)$ is called the *limit constant for* $\{\mu_n\}$ or the *limit constant associated with the moment sequence* $\{\mu_n\}$.

Associated with a general summability method, we have the following notions. Given any series $u_0+u_1+u_2+\dots$ with partial sums $s_n = u_0+u_1+\dots+u_n$, consider the series $o+u_0+u_1+\dots$ with partial sums $\bar{s}_n = o+u_0+\dots+u_{n-1}$, so that

$$\bar{s}_n = s_{n-1} \ (n \geqslant 1) \text{ and } \bar{s}_0 = 0.$$

Any method of summability, transforming Σu_n into Σv_n, or $\{s_n\}$ into $\{t_n\}$, will be said to be *translative* [1] if it has the property that, whenever $u_0+u_1+\dots$ or $\{s_n\}$, is summable to l by the matrix, then so is $o+u_0+u_1+\dots$ or $\{\bar{s}_n\}$, and conversely. The method is said to be *translative to the left* or *translative to the right* according as the first half alone of the condition of translativity or the second (converse) half alone of the condition is fulfilled.

The problem of translativity for the Hausdorff methods has been discussed by Kuttner (1956). In Theorem I of this paper we prove a sufficient condition for the δ_0-matrix (H^*, μ_n) to be translative for the class of series with bounded partial sums and the result is similar to the one obtained by Kuttner for Hausdorff methods (Kuttner (1956), Theorem 2). Certain further considerations based on the proof of Theorem I enable us to deduce the translativity theorems of Vermes (1949, Theorems 3.III and 3.IV) for the method A (p) of Taylor series continuation, where the boundedness of the partial sums of the series is not required. Theorem II is an unpublished result of Kuttner's [2] and shows how a result following from Theorem I (stated in Remark iv) for bounded sequences can be improved.

2. Lemmas. In the proofs of our theorems we require the following lemmas:

Lemma 1: *The quasi-Hausdorff matrix* (H^*, μ_n) *is a* δ_0-*matrix if, and only if,* $\{\mu_n\}$ *is a semi-regular moment constant.*

The proof of Lemma 1 follows easily from that of a known theorem (Ramanujan (1953), Theorem 2).

Vermes (1951) has shown that if $H \equiv (h_{nk})$ is a δ_0-matrix (or an α_0-matrix) then the matrix $A \equiv (a_{nk})$ defined by

$$\left.\begin{aligned} a_{nk} &= g_{nk}-g_{n,\,k+1} \\ \text{where } g_{nk} &= h_{0k}+h_{1k}+\cdots+h_{nk} \end{aligned}\right\} \quad \dots \quad \dots \quad \dots \quad (5)$$

is a K-matrix (or a T-matrix).

Moreover the matrices H and A defined above are so related that if H defines the transformation of Σu_n, with partial sums $\{s_n\}$, into Σv_n, with partial sums

[1] These terms were introduced by Hill (1942).

[2] The author is greatly indebted to Dr. Kuttner who was kind enough to scrutinize the original draft of this paper and communicate to him (the author) his unpublished result.

$\{t_n\}$, then A defines the transformation of $\{s_n\}$ into $\{t_n\}$. In this case we say that A is the K-matrix corresponding to the δ_0-matrix H.

This notion of correspondence is used in the next lemma, which follows from the proof of a known theorem (Ramanujan (1953), Theorem 2).

LEMMA 2: If $\lambda^* \equiv (H^*, \mu_n)$ is a δ_0-*matrix, then* (H^*, μ_{n+1}) *is the K-matrix corresponding to it.*

That the above K-matrix (H^*, μ_{n+1}) is multiplicative is easily verified.

Let the matrix A define a sequence-to-sequence transformation as in (2). Let

$$z_n' = \sum_{k=0}^{\infty} a_{nk}s_{k+1} \text{ and } z_n'' = \sum_{k=0}^{\infty} a_{n,\,k+1}s_{k+1}$$

Then A is said to be *absolutely regular in the sense of Cooke* if $\left|z_n' - z_n''\right| \to 0$ as $n \to \infty$; i.e. if the limit points of $\{z_n'\}$ and $\{z_n''\}$ are identical. *Thus every method which is absolutely regular in the sense of Cooke is necessarily translative.*

With this definition we have the following

LEMMA 3: *A necessary and sufficient condition that the multiplicative method A should be absolutely regular in the sense of Cooke for bounded sequences is that*

$$\sum_{k=0}^{\infty} |a_{nk} - a_{n,\,k+1}| \to 0, \text{ as } n \to \infty \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

Cooke (1950) gives the proof of Lemma 3 only for T-matrices A, but it is easily seen that it holds for multiplicative matrices as well.

Lorentz (1948) defines a bounded sequence all of whose Banach limits[3] are equal to be an *almost convergent sequence*. A method of summability which sums all almost convergent sequences is said to be *strongly regular*. Using this definition of strong regularity, Lorentz (1948) proved that the condition (6) above is necessary and sufficient for a T-matrix A to be strongly regular.

Making use of this result of Lorentz, the present author has proved the following lemma (Ramanujan (1957), Theorem 9).

LEMMA 4: *The T-matrix* (H^*, μ_{n+1}) *is strongly regular if, and only if,* $\mu_n \to 0$, *or equivalently, the function* $\chi(t)$ *generating the moment constants* $\{\mu_n\}$ *is such that* $\chi(1) = \chi(1-0)$.

An immediate consequence of Lemma 4 is

LEMMA 5: *The method* $A(p)$ *of Taylor series continuation is strongly regular for* $0 < p < 1$.

Another result which we require is

LEMMA 6: *Let the matrix A of* (2) *be a T-matrix and such that*

$$\varliminf_{n \to \infty} \left\{ |a_{nn}| - \sum_k{}^{*} |a_{nk}| \right\} > 0 \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

where the star on the Σ *signifies that the term for which* $k = n$ *is to be omitted from the sum. Then, for bounded sequences* $\{s_n\}$, *the convergence to l of* $\{t_n\}$ *implies the convergence of* $\{s_n\}$ *also to l.*

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[3] For definition of 'Banach Limits' see, for example, Banach (1932), p. 33.

The above lemma was originally proved by Agnew (1952) and an alternative proof of the same may be found in Parameswaran (to be published in *Proc. Amer. Math. Soc.*).

Two more lemmas which we require are the following:

LEMMA 7: *If* $\{\mu_n\}$ *and* $\left\{\frac{\mu_{n+1}}{\mu_n}\right\}$ *are both semi-regular moment constants, then the limit constant associated with* $\left\{\frac{\mu_{n+1}}{\mu_n}\right\}$ *is non-zero, positive and not greater than 1.*

That the limit constant is not zero is proved by an argument used to prove a well-known theorem (Hardy (1949), Theorem 212). After this, it follows from an argument of Kuttner's (1956) proof of Lemma 5 that the limit constant is not greater than 1.

LEMMA 8: *Let*

$$G_n(t) = \sum_{m \geqslant n} \frac{1}{n+1} \binom{m}{n} (1-t)^{m-n-1} t^{n+1} \,|(n+1)-(m+1)t| \qquad \ldots \quad (8)$$

where n is a positive integer. Then

(*i*) $G_n(t)$ *is bounded for all n and* $0 < t < 1$

(*ii*) $G_n(t) \to 0$, *as* $n \to \infty$ *for every t in* $\delta \leqslant t \leqslant 1-\delta (0 < \delta < \frac{1}{2})$.

PROOF: Since the expression within the modulus in the expression (8) can be written as $(n+1)(1-t)-(m-n)t$,
we have, for $0 < t < 1$,

$$G_n(t) \leqslant \sum_{m \geqslant n} \binom{m}{n} (1-t)^{m-n} t^{n+1} + \sum_{m \geqslant n+1} \binom{m}{n+1} (1-t)^{m-n-1} t^{n+2} = 2.$$

This proves (i); to prove (ii) we observe that

$$G_n(t) = \sum_{m \geqslant n} \left| \binom{m}{n} (1-t)^{m-n-1} t^{n+1} - \binom{m+1}{n+1} (1-t)^{m-n-1} t^{n+2} \right|$$

$$= \sum_{m \geqslant n} \left| \left\{ \binom{m+1}{n+1} - \binom{m}{n+1} \right\} (1-t)^{m-n-1} t^{n+1} - \binom{m+1}{n+1} (1-t)^{m-n-1} t^{n+2} \right|$$

$$= \sum_{m \geqslant n} \left| \binom{m}{n+1} (1-t)^{m-n-1} t^{n+1} - \binom{m+1}{n+1} (1-t)^{m-n} t^{n+1} \right|$$

and so,

$$tG_n(t) = \sum_{m \geqslant n} \left| \binom{m}{n+1} (1-t)^{m-n-1} t^{n+2} - \binom{m+1}{n+1} (1-t)^{m-n} t^{n+2} \right|.$$

Consider now the matrix $A \equiv a_{nk}(t)$ defined by

$$a_{nk}(t) = \begin{cases} \binom{k}{n} (1-t)^{k-n} t^{n+1}, & k \geqslant n \\ 0 & , k < n \end{cases}$$

This is the matrix of the method $A(t)$ of Taylor series continuation (of Vermes, 1949) in its sequence-to-sequence form and is a quasi-Hausdorff matrix (H^*, μ_{n+1}) with $\mu_n = t^n$ which is a T-matrix for $0 < t < 1$. Now, by Lemma 5, A is strongly regular for $0 < t < 1$ and, therefore, by Lemma 4,

$$\sum_{m \geqslant n} \left| a_{nm}(t) - a_{n,\, m+1}(t) \right| \to 0, \text{ as } n \to \infty .$$

Thus,

$$tG_n(t) \to 0, \text{ for every } t \text{ in } \delta \leqslant t \leqslant 1-\delta,\ 0 < \delta < \tfrac{1}{2},$$

and consequently the same is true of $G_n(t)$. This completes the proof of Lemma 8.

3. Theorems.

In what follows we assume that none of the μ_n vanishes.

Theorem I: *Let (H^*, μ_n) and $\left(H^*, \dfrac{\mu_{n+1}}{\mu_n}\right)$ be both δ_0-matrices. Then, for the class* **B** *of series with bounded partial sums, (H^*, μ_n) is translative to the left. If, in addition, the limit constant associated with $\left\{\dfrac{\mu_{n+1}}{\mu_n}\right\}$ is not $\frac{1}{2}$, then (H^*, μ_n) is translative, again for the class* **B**.

Proof: Let $\{s_n\}$ denote the sequence of partial sums of $\sum u_n$. By Lemma 2, it is enough to consider the translativity properties of (H^*, μ_{n+1}) which is a multiplicative matrix, since (H^*, μ_n) is a δ_0-matrix. We have that $\{\mu_n\}$ and $\left\{\dfrac{\mu_{n+1}}{\mu_n}\right\}$ are both semi-regular moment constants.

Let $\{\bar{s}_n\}$ denote as usual the sequence

$$\bar{s}_n = s_{n-1} (n \geqslant 1),\ \bar{s}_0 = 0.$$

Let (H^*, μ_{n+1}) transform $\{s_n\}$ into $\{t_n\}$ and $\{\bar{s}_n\}$ into $\{\bar{t}_n\}$.
For simplicity, let λ^*[4] denote in this proof only the matrix of (H^*, μ_{n+1}) transformation, so that

$$\lambda^*_{nk} = \binom{k}{n} \Delta^{k-n} \mu_{n+1},\ (k \geqslant n) \text{ and } \lambda^*_{nk} = 0 (k < n).$$

Let λ^{*-1} denote the reciprocal of λ^*. Then $\lambda^{*-1} = \left(H^*, \dfrac{1}{\mu_{n+1}}\right)$.

Let B denote the matrix defined by

$$b_{0n} = 0;\ b_{n,\, n-1} = 1,\ b_{nr} = 0\ (r \neq n-1),\ n \geqslant 1.$$

Then

$$\bar{s} = Bs.$$

Consider now the transformation of $\{t_n\}$ by the matrix $(\lambda^*.B\lambda^{*-1})$ which we shall assume to be a K-matrix, justifying this assumption later. We have, with this assumption,

$$(\lambda^*.\, B\lambda^{*-1})t = (\lambda^*.\, B\lambda^{*-1})(\lambda^* s) = [(\lambda^*.\, B\lambda^{*-1}).\, \lambda^*]s \ldots\ldots (\text{I})$$

[4] Note the slightly different meaning attached to λ^* in the context of (3).

since $s_n = O(1)$ and therefore (s) above is a K-matrix and the product of K-matrices is associative. Now,

$$[(\lambda^*.B\lambda^{*-1}).\lambda^*]_{nm} = \sum_k (\lambda^*.B\lambda^{*-1})_{nk}.\lambda^*_{km}$$

$$= \sum_{k=n-1}^{m} \left\{ \sum_{i=n}^{k+1} \lambda^*_{ni}.\lambda^{*-1}_{i-1,k} \right\} \lambda^*_{km}$$

$$= \sum_{i=n}^{m+1} \lambda^*_{ni} \sum_{k=i-1}^{m} \lambda^{*-1}_{i-1,k}.\lambda^*_{km}$$

$$= \sum_i \lambda^*_{ni}\, b_{im} = (\lambda^* B)_{nm}.$$

Therefore we have

$$(\lambda^*.B\lambda^{*-1})t = (\lambda^* B)s = (\lambda^*)Bs = \bar{t}.$$

It follows that if $\{s_n\}$ is bounded and $(\lambda^*.B\lambda^{*-1})$ is a K-matrix, then $\{\bar{t}_n\}$ is the transform of $\{t_n\}$ by $(\lambda^*.B\lambda^{*-1})$. We shall now prove that, under the hypothesis of the theorem, $A \equiv (\lambda^*.B\lambda^{*-1})$ is a T-matrix. The elements of this matrix A are given by

$$a_{nm} = \sum_i \lambda^*_{ni}.\lambda^{*-1}_{i-1,m}$$

$$= \sum_{i=n}^{m+1} \binom{i}{n} \Delta^{i-n}\mu_{n+1}.\binom{m}{i-1} \Delta^{m-\overline{i-1}} \left(\frac{1}{\mu_i}\right),$$

so that

$$a_{n+1,m} = \sum_{i=n+1}^{m+1} \binom{i}{n+1} \Delta^{i-\overline{n+1}} \mu_{n+2}.\binom{m}{i-1} \Delta^{m-\overline{i-1}} \left(\frac{1}{\mu_i}\right)$$

$$= \sum_{I=i-1=n}^{m} \binom{I+1}{n+1} \Delta^{I-n}\mu_{n+2}.\binom{m}{I} \Delta^{m-I} \left(\frac{1}{\mu_{I+1}}\right).$$

Putting $I = n+p$ and using the relation

$$\binom{m}{n+p}\binom{n+p+1}{n+1} = \frac{1}{n+1}\binom{m}{n}\binom{m-n}{p}(n+p+1)$$

$$= \frac{1}{n+1}\binom{m}{n}\left\{(n+1)\binom{m-n}{p}+(m-n)\binom{m-n-1}{p-1}\right\},$$

we get

$$a_{n+1,m} = \frac{1}{n+1}\binom{m}{n}\left[(n+1)\sum_{p=0}^{m-n}\binom{m-n}{p}\Delta^p\mu_{n+2}.\Delta^{m-n-p}\left(\frac{1}{\mu_{n+p+1}}\right)\right.$$

$$\left. +(m-n)\sum_{P=p-1=0}^{m-n-1}\binom{m-n-1}{P}\Delta^{P+1}\mu_{n+2}.\Delta^{m-n-P-1}\left(\frac{1}{\mu_{n+P+2}}\right)\right]$$

Applying now the well-known formula [5]

$$\Delta^N c_m d_m = \sum_{p=0}^{N} \binom{N}{p} \Delta^{N-p} c_{m+p} \Delta^p d_m,$$

we obtain

$$a_{n+1,m} = \frac{1}{n+1}\binom{m}{n}\left\{(n+1)\,\Delta^{m-n}\left(\frac{\mu_{n+2}}{\mu_{n+1}}\right)+(m-n)\,\Delta^{m-n-1}(1)\right.$$
$$\left.-(m-n)\,\Delta^{m-n-1}\left(\frac{\mu_{n+3}}{\mu_{n+2}}\right)\right\}. \quad .. \quad (9)$$

The middle term above vanishes for $m \neq n+1$. It is also easily seen that

$$a_{0m} = \Delta^m(1) - \Delta^m\left(\frac{\mu_2}{\mu_1}\right), \sum_m |a_{0m}| < \infty. \quad .. \quad .. \quad (10)$$

Since now $\left(H^*, \frac{\mu_{n+1}}{\mu_n}\right)$ is by hypothesis a δ_0-matrix, Lemma 1 shows that $\left\{\frac{\mu_{n+1}}{\mu_n}\right\}$ is a semi-regular moment constant and that consequently there exists a function $\beta(t)$ of bounded variation in (0,1) such that $\beta(+0) = \beta(0) = 0$ and

$$\frac{\mu_{n+1}}{\mu_n} = \int_0^1 t^n d\beta(t) = \int_0^{1-0} t^n d\beta(t) + k$$

where $k = \beta(1) - \beta(1-0)$ is the limit constant associated with the moment sequence $\left\{\frac{\mu_{n+1}}{\mu_n}\right\}$.

Now

$$\Delta^p\left(\frac{\mu_{n+1}}{\mu_n}\right) = \int_0^{1-0} t^n(1-t)^p d\beta(t),\ p > 1, .. \quad .. \quad .. \quad (11)$$

the case $p = 0$ requiring the addition of k on the right of (11). If we write

$$c_{n+1,m} = \frac{1}{n+1}\binom{m}{n}\int_0^{1-0} t^{n+1}(1-t)^{m-n-1}[(n+1)-(m+1)t]d\beta(t)$$
$$= \frac{1}{n+1}\binom{m}{n}\int_0^{1-0} t^{n+1}(1-t)^{m-n-1}[(n+1)(1-t)-(m-n)t]d\beta(t), \quad .. \quad (12)$$

we shall have from (9) and (11)

$$a_{n+1,n+1} = c_{n+1,n+1} + 1 - k,$$
$$a_{n+1,n} = c_{n+1,n} + k,$$
$$a_{n+1,m} = c_{n+1,m},\ m > n+2,$$
$$a_{n+1,m} = 0 = c_{n+1,m},\ m < n-1.$$

Therefore

$$\sum_m |a_{n+1,m}| < \sum_m |c_{n+1,m}| + |k| + |1-k| = \sum_m |c_{n+1,m}| + 1,$$

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[5] *See*, for example, Boole (1880), pp. 20-21.

since, by Lemma 7, $0 < k < 1$. Now, from (12) and (8),

$$\sum_m |c_{n+1,m}| < \int_0^{1-0} G_n(t)\,|\,d\beta(t)\,|.$$

Since $G_n(t) \to 0$ for every t in $0 < t < 1$ and the function $\beta(t)$ is such that $\beta(+0) = \beta(0) = 0$ and $G_n(t)$ is bounded at $t = 0$ also, we have, by Lebesgue's theorem on dominated convergence, that

$$\sum_m |c_{n+1,m}| \to 0, \text{ as } n \to \infty.$$

Also,

$$\sum_m a_{n+1,m} \to 1 \text{ as } n \to \infty \text{ and } \lim_n a_{n+1,m} = 0.$$

Thus $A \equiv (a_{nm})$ is a T-matrix and this proves the first part of the theorem.

To prove the second part, consider

$$\left\{ |a_{n+1,n+1}| - \sum_m{}^{*} |a_{n+1,m}| \right\} = \left\{ |c_{n+1,n+1}+1-k| - |c_{n+1,n}+k| + \right.$$

$$\left. + \sum_{m \geq n+2} |c_{n+1,m}| \right\}.$$

Since $\sum_m |c_{n+1,m}| \to 0$ as $n \to \infty$ and therefore $c_{n+1,m} \to 0$ as $n \to \infty$ for each m, we have

$$\lim_{n\to\infty} \left\{ |a_{n+1,n+1}| - \sum_m{}^{*} |a_{n+1,m}| \right\} = (1-k)-k = 1-2k > 0 \text{ if } k < \tfrac{1}{2},$$

and therefore, by Lemma 6, we have that λ^* is translative to the right if $k < \frac{1}{2}$. If $k > \frac{1}{2}$, consider the matrix A' defined by

$$a'_{nm} = a_{n,m-1},\ m = 1, 2 \ldots . \text{ and } a'_{n0} = 0.$$

Then

$$\bar{t}_n = \sum_{m=0}^{\infty} a_{nm} t_m = \sum_{m=0}^{\infty} a'_{n,m+1} t_m = \sum_{m=0}^{\infty} a'_{nm} t'_m, \text{ where } t'_m = t_{m-1},\ m \geq 1,\ t'_0 = 0.$$

(II) The sequence $\{t_n\}$ is bounded since $\{s_n\}$ is bounded, and $t = \lambda^* s$ where λ^* is a K-matrix.

The same property is true of $\{t'_n\}$ also; and the convergence of $\{t_n\}$ is equivalent to that of $\{t'_n\}$. Now, as before,

$$\lim_{n\to 0} \left\{ |a'_{n+1,m+1}| - \sum_m{}^{*} |a'_{n+1,m}| \right\} = k-(1-k) = 2k-1 > 0 \text{ since } k > \tfrac{1}{2}$$

and therefore, again by Lemma 6, we have that the method λ^* is translative to the right. Thus finally λ^* is translative to the right if $k \neq \frac{1}{2}$ and this completes the proof of the theorem.

REMARKS ON THEOREM I. (i) We have made use of the boundedness of the sequence $\{s_n\}$ at the two places marked (I) and (II). At (I) we use the boundedness to prove the associativity. If now, in addition to the hypothesis of the theorem $(\lambda^*. B\lambda^{*-1})$ is row-finite (so that it is a row-finite K-matrix) the associativity mentioned above is true even without the boundedness of the sequence $\{s_n\}$. If the row-finite matrix $(\lambda^*. B\lambda^{*-1})$ is even more restricted in the sense that it is a lower semi-matrix, then the proof of the theorem for the translativity to the right holds good in the case $k < \frac{1}{2}$ even for unbounded sequences, by virtue of Agnew's theorem for triangular matrices (Agnew (1952), Theorem 1.4).

The relaxation of the boundedness of $\{s_n\}$ to the above types of matrices, though admittedly not of great importance, is by no means hypothetical as is seen in the case of the method A (p) of Taylor series continuation of Vermes (1949). In this case $\lambda^*. B\lambda^{*-1}$ is triangular and is defined as below,

$$(\lambda^* . B\lambda^{*-1})_{n, n-1} = p, \quad (\lambda^* . B\lambda^{*-1})_{n, n} = 1-p, \quad (\lambda^* . B\lambda^{*-1})_{n, m} = 0, \quad (m \neq n, n-1),$$

and thus, in the light of the above remark, *the proof of Theorems 3.III and 3.IV of Vermes (1949) is included in that of our theorem.*

(ii) At the second place marked (II), we have made use of the boundedness of $\{s_n\}$ to prove the boundedness of $\{t_n\}$ and that of $\{t_n'\}$. Suppose the matrix $(\lambda^* . B\lambda^{*-1})$ is a normal triangular matrix as indicated above; then, in proving the translativity of the method for $k > \frac{1}{2}$, it is certainly sufficient for the application of Lemma 6 to assume that $\{t_n\}$ is bounded instead of assuming that $\{s_n\}$ is bounded. *Thus, in addition to Theorems 3.III and 3.IV of Vermes (1949), we see that if the method $A(p)$ is such that the transformation of $\{s_n\}$ is bounded, then for $p > \frac{1}{2}$ also the method is translative to the right.*

(iii) Considering now the case $k = \frac{1}{2}$ for the method $A(p)$, we have, in the notation of Theorem I,

$$\bar{t}_n = \frac{t_n + t_{n-1}}{2}.$$

In case we do not assume the boundedness of $\{s_n\}$, we cannot prove the convergence of $\{t_n\}$ assuming that of $\{\bar{t}_n\}$.

(iv) In Theorem I, let $s_n = O(1)$ and $k < 1$. Then $\mu_n \to 0$, $(n \to \infty)$. Hence, by Lemma 4, if $\lambda^* \equiv (H^*, \mu_{n+1})$ in Theorem I is a T-matrix, then it is a strongly regular T-matrix and therefore, by a theorem of Lorentz (1948), satisfies the condition (6) and is consequently translative. This conclusion does not necessarily follow when $k = 1$ and μ_n does not necessarily tend to 0.

We give below Kuttner's theorem, referred to in the introduction, which covers the case $k = \frac{1}{2}$ excluded in Theorem I as well as the case $k = 1$ excluded in Remark (iv) above.

THEOREM II: *If (H^*, μ_{n+1}) is any conservative sequence-to-sequence quasi-Hausdorff transformation, then (H^*, μ_{n+1}) is translative for bounded sequences.*

PROOF: Let

$$\bar{G}_n(t) = \frac{1}{n+1} t^{n+1} \sum_{m \geqslant n} \binom{m}{n} (1-t)^{m-n} \mid (n+1)-(m+1)t \mid . \quad .. \quad (13)$$

Then we shall first prove that

$$\bar{G}_n(t) \text{ is bounded for all } n \text{ and } 0 < t < 1. \quad .. \quad .. \quad (14)$$

$$\bar{G}_n(t) \to 0 \text{ as } n \to \infty \text{ for every } t \text{ in } \delta < t < 1, \ (\delta > 0). \quad .. \quad (15)$$

The proof of (14) follows from the proof of (i) of Lemma 8. To prove (15), we note that, from (13) and (8),

$$\bar{G}_n(t) = (1-t)G_n(t).$$

But $G_n(t) \to 0$ for t in $0 < \delta \leqslant t < 1$ and therefore the same is true of $\bar{G}_n(t)$. Since (13) shows that $\bar{G}_n(1) = 0$, the proof of (14) is complete.

Now, as in Theorem I, let $\{t_n\}$ denote the transform of $\{s_n\}$ and $\{\bar{t}_n\}$ that of $\{\bar{s}_n\}$. By hypothesis $\{s_n\}$ is bounded and it is clearly sufficient to show that

$$|t_n - \bar{t}_{n+1}| \to 0, \text{ as } n \to \infty$$

where

$$t_n = \sum_{m \geqslant n} \binom{m}{n} \Delta^{m-n} \mu_{n+1} s_m \quad \text{and} \quad \bar{t}_{n+1} = \sum_{m \geqslant n} \binom{m+1}{n+1} \Delta^{m-n} \mu_{n+2} s_m$$

i.e. it is sufficient to show that

$$\sum_{m \geqslant n} \binom{m}{n} \left| \Delta^{m-n}\mu_{n+1} - \left[\frac{m+1}{n+1}\right] \Delta^{m-n}\mu_{n+2} \right| \to 0, \text{ as } n \to \infty. \quad \text{.. (16)}$$

Now, by hypothesis, there is a function $\chi(t)$ of bounded variation in (0, 1), such that

$$\mu_n = \int_0^1 t^n d\chi(t).$$

Since a change in the value of $\chi(0)$ will alter μ_0 only, and not any of the other μ_n's, and since the (H^*, μ_{n+1}) matrix does not involve μ_0, there is no loss of generality in assuming that $\chi(+0) = \chi(0) = 0$. Thus (16) can be written in the form

$$\sum_{m \geqslant n} \binom{m}{n} \int_0^1 (1-t)^{m-n} t^{n+1} \left(1 - \frac{m+1}{n+1} t\right) d\chi(t) \to 0 \text{ as } n \to \infty. \quad \text{.. (17)}$$

The absolute value of the expression on the left of (17) does not exceed

$$\int_0^1 t^{n+1} \sum_{m \geqslant n} \binom{m}{n} (1-t)^{m-n} \left| 1 - \frac{m+1}{n+1} t \right| |d\chi(t)|.$$

The result now follows from (14), (15) and an application of Lebesgue's theorem on dominated convergence as in the proof of Theorem I.

Theorem I in contrast to Theorem II is of some interest in that its proof can be adapted to deduce the results of Vermes already referred to, for general (unrestricted) sequences. Since the translativity of quasi-Hausdorff methods for such sequences remains an open question, the possibility opened out by the proof of Theorem I, and indicated in the remarks following the proof, may not be altogether devoid of interest.

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