AN EXACT SOLUTION OF A SPHERICAL BLAST WAVE UNDER TERRESTRIAL CONDITIONS

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ABSTRACT

An exact solution of the equations of one-dimensional motion of a gas representing a spherical wave of explosion headed by shock has been obtained in this paper. The shock moves with constant velocity and advances in a quiescent atmosphere of constant pressure and density. The solution is applicable to both weak and strong shocks.

1. INTRODUCTION

A spherical wave of explosion under terrestrial conditions with a shock surface as wave front has been studied by G. I. Taylor (1950). Taylor reduced the equations governing the flow behind the shock wave into ordinary differential equations which were integrated numerically. His solutions, however, are applicable only to very intense explosions for which the Mach number at the head of the shock surface may be taken as very large. The total energy within the expanding wave was taken to be constant. J. L. Taylor (1955) succeeded in obtaining an analytic solution of the equations of G. I. Taylor's problem for large Mach number. Z. Kopal *et al.* (1951*a*, *b*) have also investigated the advance of spherical waves headed by shock in stellar bodies. The problem was reduced to one of numerical integrations of ordinary differential equations.

In the present paper, by a combination of the methods of Z. Kopal *et al.* and of J. L. Taylor it has been possible to obtain an exact solution of the equations for an explosion under terrestrial conditions, namely, when the wave advances into a region of constant density and pressure. The solution is applicable to both weak and strong shocks. The shock front is supposed to be moving with constant velocity. Constancy of the total energy behind the shock has not been assumed and in fact is not satisfied. Besides only particle isentropy for motion behind the shock has been assumed.

2. EQUATIONS OF THE PROBLEM AND BOUNDARY CONDITIONS

As equations governing the flow behind a spherical shock are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r}=0 \quad \ldots \quad (1)$$

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial r}+\rho\left(\frac{\partial u}{\partial r}+\frac{2u}{r}\right)=0, \quad \ldots \quad (2)$$

and further assuming isentropy for each element of fluid

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^{\gamma}}\right)+u\frac{\partial}{\partial r}\left(\frac{p}{\rho^{\gamma}}\right)=0, \quad \ldots \quad (3)$$

where u, p and ρ are velocity, pressure and density of the gas at radial distance r from the centre at time t; γ is the ratio of the specific heats. No gravitational force is considered.

This motion will be supposed to be bounded on the outside by a shock surface at $r = R(t)$, which will move outward with velocity

$$V = \frac{dR}{dt}.$$

If ahead of the shock the undisturbed pressure and density be p_0, ρ_0, and those just behind the shock p_1, ρ_1, then we have the following Rankine-Hugoniot conditions at the shock surface.

$$\frac{\rho_1}{\rho_0} = \frac{(\gamma-1)+(\gamma+1)y}{(\gamma+1)+(\gamma-1)y};\ \frac{V-u_1}{V} = \frac{\rho_0}{\rho_1};\ \rho_1 V u_1 = (y-1)p_0, \quad \ldots \quad (4, 5, 6)$$

where
$$y = \frac{p_1}{p_0}.$$

From (4), (5) and (6) we get

$$u_1 = \frac{2V}{\gamma+1} - \frac{2\gamma p_0}{(\gamma+1)\rho_0 V}, \quad \ldots \quad (7)$$

$$p_1 = \frac{2\rho_0 V^2-(\gamma-1)p_0}{\gamma+1}, \quad \ldots \quad (8)$$

$$\rho_1 = \frac{(\gamma+1)\rho_0^2\, V^2}{(\gamma-1)\rho_0\, V^2+2\gamma p_0} \quad \ldots \quad (9)$$

Next let us seek solution of equations (1), (2) and (3) in the form

$$u = \frac{r}{t}\, U(\eta);\ p = r^{K+2}t^{\lambda-2}P(\eta);\ \rho = r^K t^\lambda \Omega(\eta) \quad \ldots \quad (10, 11, 12)$$

where

$$\eta = r^a t^b \quad (13)$$

The constants K, λ, a and b are for the present kept open and are to be determined from the conditions of the problem.

We choose the shock surface to be given by

$$\eta_0 = At^\mu,$$

where A and μ are constants. This choice fixes the velocity of the shock surface as

$$V = \frac{\mu-b}{a}\cdot\frac{R}{t}. \quad \ldots \quad (14)$$

Let us now suppose that the shock surface advances with a constant velocity, and without any loss of generality we may take $b = 1$. Hence it follows from (14) that

$$a = \mu-1.$$

Thus we have

$$V = \frac{R}{t} = A^{\frac{1}{\mu-1}} \quad \ldots \quad (15)$$

3. Solution of Equations (1), (2) and of (3)

The condition inside the wave will be obtained from the solution of the equations (1), (2) and (3).

From equation (12) we get by differentiation

$$\frac{\partial \rho}{\partial t} = \frac{\lambda \rho}{t} + \frac{b}{t} \cdot r^K t^\lambda \eta \Omega'(\eta) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

and

$$\frac{\partial \rho}{\partial r} = \frac{K\rho}{r} + \frac{a}{r} \cdot r^K t^\lambda \eta \Omega'(\eta). \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

Eliminating $\Omega'(\eta)$ and t, from (16), (17) and (15) we get

$$\frac{\partial \rho}{\partial t} = \lambda \rho \frac{V}{R} + \frac{b}{a} \cdot \frac{r}{R} \cdot V \left(\frac{\partial \rho}{\partial r} - \frac{K\rho}{r} \right). \qquad \ldots \quad \ldots \quad \ldots \quad (18)$$

Treating equation (11) in the same way we get

$$\frac{\partial p}{\partial t} = (\lambda - 2) p \frac{V}{R} + \frac{b}{a} \cdot \frac{r}{R} \cdot V \left(\frac{\partial p}{\partial r} - (K+2) \frac{p}{r} \right). \qquad \ldots \quad \ldots \quad (19)$$

From equations (1), (2) and (3) it can be shown that

$$\frac{\partial E}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u I) = 0, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

where

$$E = \frac{1}{2} \rho u^2 + \frac{p}{\gamma - 1} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

and

$$I = \frac{1}{2} \rho u^2 + \frac{\gamma p}{\gamma - 1}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

Now we find

$$E = t^{\lambda - 2 - \frac{b}{a}(K+2)} \Phi(\eta), \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

where $\Phi(\eta)$ is a function of η, expressible in terms of $U(\eta)$, $P(\eta)$ and $\Omega(\eta)$.

From equation (23) we get

$$\frac{\partial E}{\partial t} = \left\{ \lambda - 2 - \frac{b}{a}(K+2) \right\} \cdot \frac{E}{t} + \frac{b}{a} \cdot \frac{r}{t} \cdot \frac{\partial E}{\partial r}. \qquad \ldots \quad \ldots \quad (24)$$

Let us now choose the constants such that

$$\lambda - 2 - \frac{b}{a}(K+2) = 3 \frac{b}{a}$$

so that

$$\frac{b}{a} = \frac{\lambda - 2}{K+5}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

Without any loss of generality we may choose

$$K = -5; \ \lambda = 2; \ b = 1; \text{ and } a = \mu - 1, \ldots \qquad \ldots \quad \ldots \quad (26)$$

so that the solutions take the form

$$\eta = \frac{t}{r^{1-\mu}}, \ u = \frac{r}{t}\, U(\eta),\ p = \frac{1}{r^3}\, P(\eta),\ \rho = \frac{t^2}{r^5}\,\Omega(\eta).$$

Let the Mach number M at the shock front be defined by

$$M^2 = \frac{V^2}{C_0^2}, \qquad \text{(27)}$$

where $C_0^2 = \frac{\gamma p_0}{\rho_0}$, is the square of the velocity of sound in the undisturbed air, since V and C_0 are constants it follows from equation (27) that M is constant.
From the Rankine-Hugoniot conditions (7), (8) and (9) we get

$$u_1 = \frac{2V}{(\gamma+1)} \cdot \left[1 - \frac{1}{M^2}\right], \qquad \text{(28)}$$

$$p_1 = \frac{p_0}{\gamma+1} \cdot \left[2\gamma M^2 - (\gamma-1)\right], \qquad \text{(29)}$$

and

$$\rho_1 = \frac{(\gamma+1)\rho_0 M^2}{(\gamma-1)M^2+2}. \qquad \text{(30)}$$

Equations (28), (29) and (30), from which also follows the constancy of u_1, p_1 and ρ_1, constitute the external boundary conditions of our problem. Next let us proceed to solve the differential equations (1), (2) and (3).

From equations (20) and (24) we have

$$\frac{\partial}{\partial r}(r^2 u I) = \frac{\partial}{\partial r}\left[\frac{1}{1-\mu} \cdot \frac{E}{t} \cdot r^3\right]$$

which gives the integral

$$r^2 u I - \frac{1}{1-\mu} \cdot \frac{E}{t} \cdot r^3 = F(t).$$

We choose $F(t)$ to be equal to zero and obtain

$$\frac{E}{I} = \frac{ut}{r} \cdot (1-\mu). \qquad \text{(31)}$$

From equation (31) it follows that $\mu < 1$. Combining equation (31) with equation (15) we get

$$\frac{u}{V}(1-\mu) = \frac{E}{I} \cdot \frac{r}{R}.$$

Now we put

$$\frac{u}{V} = u' \text{ and } \frac{r}{R} = r'$$

in equation (31) and solving for $\frac{p}{\rho}$ get

$$\frac{p}{\rho} = Cu'^2 \cdot \frac{r' - (1-\mu)u'}{\gamma u'(1-\mu) - r'}, \qquad \text{(32)}$$

where

$$C = \frac{\gamma - 1}{2} V^2.$$

Combining equations (2) and (3) we have

$$\frac{1}{p}\frac{\partial p}{\partial r}-\frac{\gamma-1}{\rho}\cdot\frac{\partial \rho}{\partial r}=-\frac{1}{pu}\cdot\frac{\partial p}{\partial t}+\frac{\gamma-1}{\rho u}\cdot\frac{\partial \rho}{\partial t}-\frac{1}{u}\cdot\frac{\partial u}{\partial r}-\frac{2}{r}. \qquad (33)$$

Replacing $\frac{\partial p}{\partial t}$, and $\frac{\partial \rho}{\partial t}$ on the right-hand side by means of equations (18) and (19) we get

$$\frac{1}{p}\cdot\frac{\partial p}{\partial r}-\frac{\gamma-1}{\rho}\cdot\frac{\partial \rho}{\partial r}=-\frac{2}{r}-\frac{(\gamma-1)(2\mu+3)}{1-\mu}\cdot\frac{\frac{1}{R}}{\frac{u}{V}-\frac{1}{1-\mu}\cdot\frac{r}{R}}-\frac{\frac{1}{V}\cdot\frac{\partial u}{\partial r}-\frac{1}{1-\mu}\cdot\frac{1}{R}}{\frac{u}{V}-\frac{1}{1-\mu}\cdot\frac{r}{R}}. \qquad (34)$$

Equation (34) can be integrated with respect to r, in fact integration gives

$$\frac{p}{\rho^{\gamma-1}}=C'r'^{-2}(r'-u'(1-\mu))^{-1}[f(r')]^{-1}, \qquad (35)$$

where C' is a function of the time, and

$$\log f(r')=-(2\mu+3)(\gamma-1)\int_1^{r'}\frac{dr'}{r'-u'(1-\mu)}. \qquad (35a)$$

Solving for ρ and p from equations (32) and (35) we get

$$\frac{\rho}{\rho_1}=C_1(r'-u'(1-\mu))^{-\frac{2}{2-\gamma}}(u'(1-\mu)r')^{-\frac{2}{2-\gamma}}(\gamma u'(1-\mu)-r')^{\frac{1}{2-\gamma}}[f(r')]^{-\frac{1}{2-\gamma}} \qquad (36)$$

and

$$\frac{p}{p_1}=C_2(r'-u'(1-\mu))^{-\frac{\gamma}{2-\gamma}}(\gamma u'(1-\mu)-r')^{\frac{\gamma-1}{2-\gamma}}r'^{-\frac{2}{2-\gamma}}(u'(1-\mu))^{\frac{2-2\gamma}{2-\gamma}}[f(r')]^{-\frac{1}{2-\gamma}}, \qquad (37)$$

where C_1 and C_2 are determined from the value of u_1 as given by the Rankine-Hugoniot equation (28). We have in fact

$$\left.\begin{aligned} C_1&=\left(1-(1-\mu)u_1'\right)^{\frac{2}{2-\gamma}}\left(u_1'(1-\mu)\right)^{\frac{2}{2-\gamma}}\left(\gamma u_1'(1-\mu)-1\right)^{-\frac{1}{2-\gamma}} \\ \text{and}\quad C_2&=\left(1-(1-\mu)u_1'\right)^{\frac{\gamma}{2-\gamma}}\left(\gamma u_1'(1-\mu)-1\right)^{-\frac{\gamma-1}{2-\gamma}}\left(u_1'(1-\mu)\right)^{-\frac{2-2\gamma}{2-\gamma}}\end{aligned}\right\} \qquad (38)$$

where $u_1'=\frac{u_1}{V}$.

From equations (2) and (18) we get

$$\frac{1}{\rho}\cdot\frac{\partial \rho}{\partial r'}=\frac{1}{r'-u'(1-\mu)}\cdot\frac{\partial u'(1-\mu)}{\partial r'}+\frac{2}{r'-u'(1-\mu)}\cdot\frac{u'(1-\mu)}{r'}-\frac{(2\mu+3)}{r'-u'(1-\mu)}.$$

Using the value of ρ from equation (36) in this equation we get

$$-\frac{2}{2-\gamma}\cdot\frac{1}{r'-u'(1-\mu)}\cdot\left[1-\frac{\partial u'(1-\mu)}{\partial r'}\right]-\frac{2}{2-\gamma}\left[\frac{1}{u'(1-\mu)}\cdot\frac{\partial u'(1-\mu)}{\partial r'}+\frac{1}{r'}\right]$$
$$+\frac{1}{2-\gamma}\cdot\frac{1}{\gamma u'(1-\mu)-r'}\cdot\left[\gamma\frac{\partial u'(1-\mu)}{\partial r'}-1\right]$$
$$=\frac{1}{r'-u'(1-\mu)}\cdot\frac{\partial u'(1-\mu)}{\partial r'}-\frac{2}{r'}+\frac{2}{r'-u'(1-\mu)}-\frac{(2\mu+3)}{2-\gamma}\cdot\frac{1}{r'-u'(1-\mu)}\cdot \qquad (39)$$

Now the solution of equation (39) can be written as

$$2\log u'=\chi_1\log r'+\chi_2\log\left(r'-\frac{(3\gamma-1)}{2}u'\right)+\chi_3\log\left(\gamma u'(1-\mu)-r'\right)$$
$$+2\log u_1'-\chi_2\log\left(1-\frac{(3\gamma-1)}{2}u_1'\right)-\chi_3\log\left(\gamma u_1'(1-\mu)-1\right), \quad .. \quad (40)$$

where

$$\left.\begin{aligned}\chi_1&=\frac{-4(1-\mu)(\gamma-1)}{3\gamma-1}\\ \chi_2&=\frac{\gamma^2(8\mu-1)+\gamma(-8\mu^2+12\mu+7)+4\mu-6}{(3\gamma-1)(-2\mu\gamma-\gamma+1)}\\ \chi_3&=\frac{2(\gamma-1)(1-\mu)}{-2\mu\gamma-\gamma+1}\end{aligned}\right\} \quad .. \quad .. \quad (41)$$

Equations (36), (37), (40) and (41) give the solution of our problem. They constitute a rigorous solution of the non-linear equations (1), (2) and (3) corresponding to which the shock front moves characterized by the constant velocity V, and the constant Mach number M. In the solution of the above equations there is no restriction on the value of M which may be large or small. Thus the present solution of these equations is valid for all strengths of the head shock wave. By choosing a suitable value for μ between 0 and 1 (equation (31) shows that $\mu < 1$), and a value of M greater than 1, one obtains (for given C_0) from (27), (28), (29), and (30), V, u_1, p_1 and ρ_1. Equation (28) gives u'_1. Then one evaluates the integral (35*a*) tabulating $f(r')$ for different values of r'. From this with the help of (36), (37), (38) and (40) the functions $\frac{\rho}{\rho_1}$, $\frac{p}{p_1}$ and u_1 may be calculated.

It may be noted that though at the beginning we had taken the velocity, pressure, and density behind the shock surface in the form as given by equations (10, 11, 12), we have obtained explicit functions for ρ, p and u without determining the functions $U(\eta)$, $P(\eta)$ and $\Omega(\eta)$. Further, the solution of our equations is not really dependent on our special choice of the values of K and λ. The essential point is that they should be connected by the relation (25). This one may verify by a more complicated calculation.

We can now calculate the total energy within the shock wave at any time t which is given by

$$\Psi=4\pi\int_0^R\left[\frac{1}{2}\rho u^2+\frac{p}{\gamma-1}\right]r^2\,dr.$$

If we choose a value for μ sufficiently small we may consider the total energy behind the shock wave to be very nearly constant, or increasing at a slow rate.

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