A THEOREM ON MEIJER TRANSFORM AND INFINITE INTEGRALS INVOLVING G-FUNCTION AND BESSEL FUNCTIONS

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(Communicated by R. S. Varma, F.N.I.)

(*Received November 25 ; read December 6, 1957*)

Abstract

This paper gives a theorem on Meijer's generalized Laplace transform defined by

$$\phi(p) = \sqrt{2/\pi}p \int_0^\infty (px)^{\frac{1}{2}} K_\nu(px) f(x) dx.$$

Some infinite integrals involving G-function, hypergeometric and Bessel functions are evaluated with the help of the theorem and a few operational images are obtained.

1. A function $\psi(p)$, defined by

$$\psi(p) = p \int_0^\infty e^{-px} f(x)\, dx, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

is called Laplace transform of $f(x)$, which is called its original. Meijer (1940*a*) generalized it in the form

$$\phi(p) = (2/\pi)^{\frac{1}{2}} p \int_0^\infty (px)^{\frac{1}{2}} K_\nu(px) f(x)\, dx. \ldots \quad \ldots \quad \ldots \quad (2)$$

Since

$$K_{\pm\frac{1}{2}}(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

(2) reduces to (1) when $\nu = \pm\frac{1}{2}$.

We shall call $\phi(p)$ as Meijer transform of order ν and $f(x)$ its original in this transform.

Throughout this paper (1) and (2) shall be denoted symbolically as

$$\psi(p) \doteqdot f(x) \quad \text{and} \quad \phi(p) \underset{\nu}{\overset{K}{=}} f(x) \text{ respectively.}$$

The object of this paper is to evaluate certain infinite integrals involving hypergeometric function, Bessel functions and Meijer's G-function. The method used herein is that of operational calculus. In §2 we state and prove a theorem on Meijer transform (2) and in §3 we make use of that theorem to evaluate infinite integrals.

2. Theorem. If

$$\phi(p) \underset{\nu}{\overset{K}{=}} h(x),$$

and

$$p^{2+\sigma}h(p) \overset{K}{\underset{\mu}{=}} f(x),$$

then

$$\phi(p) = \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu-\frac{1}{2}\sigma)\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu-\frac{1}{2}\sigma)\Gamma(\frac{1}{2}-\frac{1}{2}\nu+\frac{1}{2}\mu-\frac{1}{2}\sigma)\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu-\frac{1}{2}\sigma)}{\pi\Gamma(1-\sigma)} \times$$

$$2^{-\sigma-1}p^{\nu+\frac{3}{2}}\int_0^\infty t^{\sigma-\nu-\frac{1}{2}}f(t)\,{}_2F_1(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu-\tfrac{1}{2}\sigma,\ \tfrac{1}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu-\tfrac{1}{2}\sigma;\ 1-\sigma;\ 1-p^2/t^2)\,dt, \quad (4)$$

provided that the integral is convergent and Meijer transform of $f(x)$ and $h(x)$ exist.

PROOF. Since

$$p^{2+\sigma}h(p) = (2/\pi)^{\frac{1}{2}}p\int_0^\infty (pt)^{\frac{1}{2}}K_\mu(pt)f(t)\,dt,$$

we have

$$\phi(p) = \frac{2}{\pi}p\int_0^\infty (px)^{\frac{1}{2}}K_\nu(px)x^{-\sigma-1}\int_0^\infty (tx)^{\frac{1}{2}}K_\mu(tx)\,f(t)\,dt\,dx$$

$$= \frac{2}{\pi}p^{\frac{3}{2}}\int_0^\infty t^{\frac{1}{2}}f(t)\int_0^\infty x^{-\sigma}K_\nu(px)K_\mu(tx)\,dx$$

$$= \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu-\frac{1}{2}\sigma)\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu-\frac{1}{2}\sigma)\Gamma(\frac{1}{2}-\frac{1}{2}\nu+\frac{1}{2}\mu-\frac{1}{2}\sigma)\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu-\frac{1}{2}\sigma)}{\pi\Gamma(1-\sigma)} \times$$

$$p^{\nu+\frac{3}{2}}2^{-\sigma-1}\int_0^\infty t^{-\sigma-1}f(t)\,{}_2F_1(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu-\tfrac{1}{2}\sigma,\ \tfrac{1}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu-\tfrac{1}{2}\sigma\,;\ 1-\sigma\,;\ 1-p^2/t^2)\,dt,$$

by virtue of (Erdelyi, 1953*b*, p. 93)

$$\int_0^\infty K_\mu(\alpha t)K_\nu(\beta t)t^{-\rho}\,dt$$

$$= \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu-\frac{1}{2}\rho)\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu-\frac{1}{2}\rho)\Gamma(\frac{1}{2}-\frac{1}{2}\nu+\frac{1}{2}\mu-\frac{1}{2}\rho)\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu-\frac{1}{2}\rho)}{\Gamma(1-\rho)} \times$$

$$2^{-\rho-2}\alpha^{\rho-\nu-1}\beta^{\nu}\,{}_2F_1(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu-\tfrac{1}{2}\rho,\ \tfrac{1}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu-\tfrac{1}{2}\rho\,;\ 1-\rho\,;\ 1-\beta^2/\alpha^2), \quad (5)$$

where $R(\alpha+\beta) > 0$, $R(1\pm\mu\pm\nu-\rho) > 0$.

The change of the order of integration is justified by de la Vallèe Poussin's theorem for the conditions stated with (4).

It is interesting to note that the hypergeometric function in the integrand of (4) reduces to Legendre's associated function when either $\phi(p)$ or $p^{2+\sigma}h(p)$ is ordinary Laplace transform. Thus

COROLLARY 1. If

$$\phi(p) \overset{K}{\underset{\nu}{=}} h(x),$$

and

$$p^{2+\sigma}h(p) \fallingdotseq f(x),$$

then

$$\phi(p) = p^{2+\sigma}\Gamma(\tfrac{1}{2}+\nu-\sigma)\Gamma(\tfrac{1}{2}-\nu-\sigma)\int_0^\infty (x^2-1)^{\frac{1}{2}\sigma}f(px)P^{\sigma}_{-\nu-\frac{1}{2}}(x)\,dx, \quad .. \quad (6)$$

provided that the integral is convergent.

Relation (6) is easily obtained from (4) by taking $\mu = \pm\frac{1}{2}$, using the relation (Erdelyi, 1953*a*, p. 129)

$$\Gamma(1-\mu)P^{\mu}_{\nu}(x) = 2^{\mu}(x^2-1)^{-\frac{1}{2}\mu}x^{\mu+\nu}{}_2F_1(-\tfrac{1}{2}\nu-\tfrac{1}{2}\mu,\ \tfrac{1}{2}-\tfrac{1}{2}\nu-\tfrac{1}{2}\mu;\ 1-\mu;\ 1-x^{-2}), \quad .. \quad (7)$$

and applying the duplication formula for gamma functions, namely,

$$\Gamma(z)\Gamma(z+\tfrac{1}{2}) = 2^{1-2z}\pi^{\frac{1}{2}}\Gamma(2z) \quad .. \quad .. \quad .. \quad .. \quad (8)$$

COROLLARY 2. If

$$\psi(p) \doteqdot h(x),$$

and

$$p^{2+\sigma}h(p) \underset{\mu}{\overset{K}{=}} f(x),$$

then

$$\psi(p) = p^{2+\sigma}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\mu-\tfrac{1}{2}\sigma)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\mu-\tfrac{1}{2}\sigma)\int_0^\infty x^{-\sigma-2}f(p/x)(x^2-1)^{\sigma/2}P^{\sigma}_{\mu-\frac{1}{2}}(x)\,dx, \quad .. \quad (9)$$

provided that the integral is convergent.

The Corollary 2 is obtained from the theorem by taking $\nu = -\frac{1}{2}$, using the relation (Erdelyi, 1953*a*, p. 127)

$$\Gamma(1-\mu)2^{-\mu}(z^2-1)^{\mu/2}P^{\mu}_{\nu}(z) = {}_2F_1(\tfrac{1}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu,\ -\tfrac{1}{2}\nu-\tfrac{1}{2}\mu;\ 1-\mu;\ 1-z^2) \quad .. \quad (10)$$

and the formula (8).

3. In this section we illustrate the above theorem by evaluating certain infinite integrals.

(i) Starting with the result given by Meijer (1940*b*)

$$G^{40}_{04}\left(\frac{S^2}{16}\middle|\, a,\ q,\ \tfrac{1}{2}\nu,\ -\tfrac{1}{2}\nu\right) = 2^{3-a-q}S^{a+q}\int_0^\infty K_{a-q}(St)K_{\nu}(1/t)t^{a+q-1}dt, \quad .. \quad (11)$$

and using the property of G-function (Erdelyi, 1953*a*, p. 209)

$$x^{\sigma}G^{m\,n}_{p\,q}\left(x\middle|\begin{matrix}\alpha_r\\ \beta_s\end{matrix}\right) = G^{m\,n}_{p\,q}\left(x\middle|\begin{matrix}\alpha_r+\sigma\\ \beta_s+\sigma\end{matrix}\right) \quad .. \quad .. \quad .. \quad (12)$$

we get

$$f(x) = x^{l-1}K_{\lambda}(a/x)$$

$$\underset{\mu}{\overset{K}{=}} 2^{-l-2}\pi^{-\frac{1}{2}}a^{l}pG^{40}_{04}\left(\frac{a^2p^2}{16}\middle|\, \tfrac{1}{2}\mu+\tfrac{1}{4},\ -\tfrac{1}{2}\mu+\tfrac{1}{4},\ \frac{\lambda-l}{2},\ -\frac{\lambda+l}{2}\right)$$

$$= p^{2+\sigma}h(p)[R(a) > 0],$$

and hence (Erdelyi, 1954, p. 153)

$$h(x) = 2^{-l-2}\pi^{-\frac{1}{2}}a^{l}x^{-\sigma-1}G_{04}^{40}\left(\frac{a^2x^2}{16}\middle|\, \tfrac{1}{2}\mu+\tfrac{1}{4}, -\tfrac{1}{2}\mu+\tfrac{1}{4}, \frac{\lambda-l}{2}, -\frac{\lambda+l}{2}\right)$$

$$= 2^{-l-2\sigma-4}\pi^{-\frac{1}{2}}a^{l+\sigma+1}G_{04}^{40}\left(\frac{a^2x^2}{16}\middle|\, \tfrac{1}{2}\mu-\tfrac{1}{2}\sigma-\tfrac{1}{4}, -\tfrac{1}{2}\mu-\tfrac{1}{2}\sigma-\tfrac{1}{4}, \frac{\lambda-l-\sigma-1}{2}, -\frac{\lambda+l+\sigma+1}{2}\right)$$

$$\underset{\nu}{\overset{K}{=}} 2^{-l-2\sigma-3}a^{l+\sigma}\pi^{-1}pG_{42}^{24}\left(\frac{4p^2}{a^2}\middle|\begin{matrix}\tfrac{3}{4}-\tfrac{1}{2}\mu+\tfrac{1}{2}\sigma, \tfrac{3}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}\sigma, 1+\tfrac{1}{2}l+\tfrac{1}{2}\sigma-\tfrac{1}{2}\lambda, 1+\tfrac{1}{2}l+\tfrac{1}{2}\sigma+\tfrac{1}{2}\lambda\\ \tfrac{1}{4}+\tfrac{1}{2}\nu, \tfrac{1}{4}-\tfrac{1}{2}\nu\end{matrix}\right)$$

$$= \phi(p)[R(1\pm\mu\pm\nu-\sigma) > 0,\ R(p) > 0, \text{ and } R(l+\sigma\pm\nu\pm\lambda-\tfrac{1}{2}) < 0].$$

Using the above values of $f(x)$ and $\phi(p)$ in (4), substituting p/x for t, then replacing a/p by p and putting $\sigma = 1-\gamma$, $\mu = \alpha-\beta$, $l = \alpha+\beta-\rho-1$, $\nu = \alpha+\beta-\gamma$ and using (12) we get

$$\int_0^\infty x^{\rho-\frac{1}{2}}K_\lambda(px)\,{}_2F_1(\alpha, \beta; \gamma; 1-x^2)\, dx$$

$$= \frac{1}{2^2}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}G_{42}^{24}\left(\frac{4}{p^2}\middle|\begin{matrix}\tfrac{5}{4}+\tfrac{1}{2}\rho-\alpha, \tfrac{5}{4}+\tfrac{1}{2}\rho-\beta, 1-\tfrac{1}{2}\lambda, 1+\tfrac{1}{2}\lambda\\ \tfrac{1}{4}+\tfrac{1}{2}\rho, \tfrac{1}{4}-\alpha-\beta+\gamma+\tfrac{1}{2}\rho\end{matrix}\right), \quad \ldots \quad (13)$$

$R(\rho\pm\lambda+\frac{1}{2}) > 0$, $R(\gamma-\alpha-\beta) > 0$, and $R(p) > 0$.

If we put $\gamma = \beta$ in (13) we get the known result

$$\int_0^\infty x^{\rho-2\alpha-\frac{1}{2}}K_\lambda(px)\, dx$$

$$= 2^{\rho-2\alpha-\frac{3}{2}}p^{2\alpha-\rho}\Gamma(\tfrac{1}{2}\rho+\tfrac{1}{2}\lambda-\alpha+\tfrac{1}{4})\Gamma(\tfrac{1}{2}\rho-\tfrac{1}{2}\lambda-\alpha+\tfrac{1}{4}), \quad \ldots \quad (14)$$

$R(\rho-2\alpha\pm\lambda+\frac{1}{2}) > 0$ and $R(p) > 0$.

From (13) we deduce that

$$x^{\rho-1}{}_2F_1(\alpha, \beta; \gamma; 1-x^2) \underset{\lambda}{\overset{K}{=}} \pi^{-\frac{1}{2}}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\times$$

$$\times G_{42}^{24}\left(\frac{4}{p^2}\middle|\begin{matrix}\tfrac{1}{2}+\tfrac{1}{2}\rho-\alpha, \tfrac{1}{2}+\tfrac{1}{2}\rho-\beta, \tfrac{1}{4}-\tfrac{1}{2}\lambda, \tfrac{1}{4}+\tfrac{1}{2}\lambda\\ \tfrac{1}{2}\rho-\tfrac{1}{2}, \tfrac{1}{2}\rho+\gamma-\alpha-\beta-\tfrac{1}{2}\end{matrix}\right), \quad \ldots \quad (15)$$

$R(\rho\pm\lambda+\frac{1}{2}) > 0$, $R(\gamma-\alpha-\beta) > 0$ and $R(p) > 0$.

Further taking $\lambda = \pm\frac{1}{2}$, this yields

$$x^{\rho-1}{}_2F_1(\alpha, \beta; \gamma; 1-x^2) \doteqdot \pi^{-\frac{1}{2}}\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\times$$

$$G_{42}^{24}\left(\frac{4}{p^2}\middle|\begin{matrix}\tfrac{1}{2}+\tfrac{1}{2}\rho-\alpha, \tfrac{1}{2}+\tfrac{1}{2}\rho-\beta, 0, \tfrac{1}{2}\\ \tfrac{1}{2}\rho-\tfrac{1}{2}, \tfrac{1}{2}\rho+\gamma-\alpha-\beta-\tfrac{1}{2}\end{matrix}\right), \quad \ldots \quad (16)$$

$R(\rho) > 0$, $R(p) > 0$ and $R(\gamma-\alpha-\beta) > 0$.

If we take $\alpha = \frac{3}{4}+\frac{1}{2}\lambda+\frac{1}{2}\rho$, $\beta = \frac{3}{4}-\frac{1}{2}\lambda+\frac{1}{2}\rho$, $\gamma = 2+\rho$ in (13), use the expansion (10), the formula (Erdelyi, 1953*a*, p. 209)

$$G_{pq}^{mn}\left(x^{-1}\middle|\begin{matrix}\alpha_r\\ \beta_s\end{matrix}\right) = G_{qp}^{nm}\left(x\middle|\begin{matrix}1-\beta_s\\ 1-\alpha_r\end{matrix}\right) \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

and the relation (Erdelyi, 1953*a*, p. 221)

$$e^x W_{k,m}(2x) = \frac{x^{\frac12} 2^{-(k+1)} \pi^{-\frac32}}{\Gamma(\frac12+m-k)\Gamma(\frac12-m-k)} G^{42}_{24}\left(x^2 \,\middle|\, \begin{matrix} \frac14+\frac12 k,\ \frac34+\frac12 k \\ \frac12 m,\ \frac12+\frac12 m,\ -\frac12 m,\ \frac12-\frac12 m \end{matrix}\right) \quad (18)$$

we get

$$\int_0^\infty x^{\rho-\frac12} K_\lambda(px)(x^2-1)^{-\frac12-\frac12\rho} P^{-1-\rho}_{\lambda-\frac12}(x)\,dx = \pi^{\frac12} 2^{-\frac12} p^{-\frac12}[(\rho+\tfrac12)^2-\lambda^2] e^{p/2} W_{\rho,\lambda}(2p), \quad (19)$$

where $R(p) > 0$, and $R(\rho\pm\lambda+\frac12) > 0$.

(ii) Take now (15),

$$f(x) = x^{\rho-1}{}_2F_1(a, b\,;\, c\,;\, 1-x^2)$$

$$\overset{K}{\underset{\mu}{=}} \pi^{-\frac12} \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)\Gamma(b)\Gamma(c-b)} G^{24}_{42}\left(\frac{4}{p^2} \,\middle|\, \begin{matrix} \frac12+\frac12\rho-a,\ \frac12+\frac12\rho-b,\ \frac14-\frac12\mu,\ \frac14+\frac12\mu \\ \frac12\rho-\frac12,\ \frac12\rho+c-a-b-\frac12 \end{matrix}\right)$$

$$= p^{2+\sigma} h(p)\ [R(\rho\pm\mu+\tfrac12) > 0].$$

Then using (12) and (17) we get

$$h(x) = \pi^{-\frac12} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} 2^{-2-\sigma} \times$$

$$G^{42}_{24}\left(\frac{x^2}{4} \,\middle|\, \begin{matrix} \frac12-\frac12\rho-\frac12\sigma,\ \frac12-\frac12\rho-\frac12\sigma-c+a+b \\ a-\frac12\sigma-\frac12\rho-\frac12,\ b-\frac12\sigma-\frac12\rho-\frac12,\ \frac12\mu-\frac12\sigma-\frac14,\ -\frac12\mu-\frac12\sigma-\frac14 \end{matrix}\right)$$

and therefore (Erdelyi, 1954, p. 153) we get

$$\phi(p) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} \frac{p}{\pi} 2^{-\sigma-2}$$

$$G^{44}_{44}\left(p^2 \,\middle|\, \begin{matrix} 1+\frac12\rho+\frac12\sigma-a,\ 1+\frac12\rho+\frac12\sigma-b,\ \frac34+\frac12\sigma-\frac12\mu,\ \frac34+\frac12\sigma+\frac12\mu \\ \frac14+\frac12\nu,\ \frac14-\frac12\nu,\ \frac12\rho+\frac12\sigma,\ \frac12\rho+\frac12\sigma+c-a-b \end{matrix}\right),$$

$R(p) > 0$, $R(2a-\sigma-\rho\pm\nu+\frac12) > 0$, $R(2b-\sigma-\rho\pm\nu+\frac12) > 0$ and $R(1\pm\mu\pm\nu-\sigma) > 0$.

Using the above values of $f(x)$ and $\phi(p)$ in (4), putting $\mu = \alpha-\beta$, $\sigma = 1-\gamma$, $\nu = \alpha+\beta-\gamma$, $\rho = 2k+\alpha+\beta-\frac12$, applying (12), replacing p^2 by p and substituting x for t^2, we get

$$\int_0^\infty x^{k-1}{}_2F_1(a, b\,;\, c\,;\, 1-x){}_2F_1(\alpha, \beta\,;\, \gamma\,;\, 1-p/x)\,dx$$

$$= \frac{\Gamma(c)\Gamma(\gamma)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \times$$

$$G^{44}_{44}\left(p \,\middle|\, \begin{matrix} 1+k-a,\ 1+k-b,\ 1-\alpha,\ 1-\beta \\ 0,\ \gamma-\alpha-\beta,\ k,\ k-a-b+c \end{matrix}\right), \quad .. \quad (20)$$

where

$R(k+\alpha) > 0$, $R(k+\beta) > 0$, $R(a-k) > 0$, $R(b-k) > 0$, $R(c-a-b) > 0$, $R(\gamma-\alpha-\beta) > 0$ and $R(p) > 0$.

(iii) Starting with the integral (Erdelyi, 1953*a*, p. 215)

$$\int_0^\infty x^{-l}K_\nu(2\sqrt{x})G^{m\,n}_{p\,q}\left(cx\,\middle|\,\begin{matrix}a_1, a_2, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}\right)dx$$

$$= \tfrac{1}{2}\, G^{m,\,n+2}_{p+2,\,q}\left(c\,\middle|\,\begin{matrix}l-\tfrac{1}{2}\nu,\, l+\tfrac{1}{2}\nu,\, a_1, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}\right), \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

$R(1\pm\tfrac{1}{2}\nu+b_h-l) > 0\,[h = 1, 2, \ldots m\,.],\ p+q < 2(m+n),\ |\arg c| < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,$

and using (17) we get

$$x^{\frac{1}{2}-2l}G^{m\,n}_{r\,s}\left(\lambda x^2\,\middle|\,\begin{matrix}\alpha_1, \ldots, \alpha_r\\ \beta_1, \ldots, \beta_s\end{matrix}\right)$$

$$\underset{\nu}{\overset{K}{=}} 2^{\frac{1}{2}-2l}p^{2l-\frac{1}{2}}\pi^{-\frac{1}{2}}G^{n+2,\,m}_{s,\,r+2}\left(\frac{p^2}{4\lambda}\,\middle|\,\begin{matrix}1-\beta_1, \ldots, 1-\beta_s\\ 1-l+\tfrac{1}{2}\nu,\, 1-l-\tfrac{1}{2}\nu,\, 1-\alpha_1, \ldots, 1-\alpha_r\end{matrix}\right), \qquad (22)$$

$r+s < 2(m+n)$, $R(p) > 0$, $R(1-l\pm\tfrac{1}{2}\nu+\beta_h) > 0\,[h = 1, 2, \ldots m]$ and $|\arg\lambda| < (m+n-\tfrac{1}{2}r-\tfrac{1}{2}s)\pi$.

Hence we have

$$f(x) = x^{\frac{1}{2}-l}G^{mn}_{rs}\left(x^2\,\middle|\,\begin{matrix}\alpha_1, \ldots, \alpha_r\\ \beta_1, \ldots, \beta_s\end{matrix}\right)$$

$$\underset{\mu}{\overset{K}{=}} 2^{\frac{1}{2}-l}p^{l-\frac{1}{2}}\pi^{-\frac{1}{2}}G^{n+2,\,m}_{s,\,r+2}\left(\frac{p^2}{4}\,\middle|\,\begin{matrix}1-\beta_1, \ldots, 1-\beta_s\\ 1-\tfrac{1}{2}l+\tfrac{1}{2}\mu,\, 1-\tfrac{1}{2}l-\tfrac{1}{2}\mu,\, 1-\alpha_1, \ldots, 1-\alpha_r\end{matrix}\right)$$

$$= p^{2+\sigma}h(p)[R(1-\tfrac{1}{2}l\pm\tfrac{1}{2}\mu+\beta_h) > 0,\ r+s < 2(m+n),\ R(p) > 0],$$

and therefore

$$h(x) = 2^{\frac{1}{2}-l}\pi^{-\frac{1}{2}}x^{l-\sigma-5/2}G^{n+2\,m}_{s\,r+2}\left(\frac{x^2}{4}\,\middle|\,\begin{matrix}1-\beta_1, \ldots, 1-\beta_s\\ 1-\tfrac{1}{2}l+\tfrac{1}{2}\mu,\, 1-\tfrac{1}{2}l-\tfrac{1}{2}\mu,\, 1-\alpha_1, \ldots, 1-\alpha_r\end{matrix}\right)$$

$$\underset{\nu}{\overset{K}{=}} \pi^{-1}2^{-\sigma-2}p^{\sigma-l+5/2}G^{m+2\,n+2}_{r+2\,s+2}\left(p^2\,\middle|\,\begin{matrix}\tfrac{1}{2}l-\tfrac{1}{2}\mu,\, \tfrac{1}{2}l+\tfrac{1}{2}\mu,\, \alpha_1, \ldots, \alpha_r\\ \tfrac{1}{2}l-\tfrac{1}{2}\sigma+\tfrac{1}{2}\nu-\tfrac{1}{2},\, \tfrac{1}{2}l-\tfrac{1}{2}\sigma-\tfrac{1}{2}\nu-\tfrac{1}{2},\, \beta_1, \ldots, \beta_s\end{matrix}\right)$$

$$= \phi(p)[R(1\pm\mu\pm\nu-\sigma) > 0,\ R(1\pm\nu+l-\sigma+2\alpha_h) > 0\ \{h = 1, 2, \ldots n\}].$$

Using the above values of $f(x)$ and $\phi(p)$ in (4), writing $a-b = \mu$, $a+b-c = \nu$, $\sigma = 1-c$, $l = 2-a-b-2\delta$ and then replacing p^2 by p and substituting x for t^2 we get

$$\int_0^\infty x^{\delta-1}G^{mn}_{rs}\left(x\,\middle|\,\begin{matrix}\alpha_1, \ldots, \alpha_r\\ \beta_1, \ldots \beta_s\end{matrix}\right){}_2F_1(a, b\,;\, c\,;\, 1-p/x)\,dx$$

$$= p^\delta\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)\Gamma(b)\Gamma(c-b)}\,G^{m+2\,n+2}_{r+2\,s+2}\left(p\,\middle|\,\begin{matrix}1-\delta-a,\, 1-\delta-b,\, \alpha_1, \ldots, \alpha_r\\ -\delta,\, c-a-b-\delta,\, \beta_1, \ldots, \beta_s\end{matrix}\right), \qquad (23)$$

$R(\beta_h+a+\delta) > 0$, $R(\beta_h+b+\delta) > 0\ [h = 1, \ldots, m]$, $R(\alpha_j+\delta) < 0$ $[j = 1, \ldots, n]$, $r+s<2(m+n)$, $R(c-a-b) > 0$ and $R(p) > 0$.

Some particular cases of (23), which are quite interesting, are given below:

(a) Taking $m=1$, $n=r=s=2$, $\alpha_1=-\alpha$, $\alpha_2=-\beta$, $\beta_1=-1$ and $\beta_2=-\gamma$ in (23) and using the relation (Erdelyi, 1953a, p. 222)

$$x\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}G_{22}^{12}\left(x\,\middle|\,\begin{matrix}-\alpha,-\beta\\-1,-\gamma\end{matrix}\right)={}_2F_1(\alpha,\beta;\gamma;-x)\quad\ldots\quad\ldots\quad(24)$$

we have

$$\int_0^\infty x^{\delta-2}{}_2F_1(\alpha,\beta;\gamma;-x)\,{}_2F_1(a,b;c;1-p/x)\,dx$$

$$=p^\delta\frac{\Gamma(c)\,\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}G_{44}^{34}\left(p\,\middle|\,\begin{matrix}1-\delta-a,\,1-\delta-b,\,-\alpha,\,-\beta\\-\delta,\,c-a-b,\,-\gamma\end{matrix}\right),\ldots\quad(25)$$

$R(a+\delta-1)>0$, $R(b+\delta-1)>0$, $R(1+\alpha-\delta)>0$, $R(1+\beta-\delta)>0$,
$R(p)>0$, and $R(c-a-b)>0$.

(b) By virtue of the relation (Erdelyi, 1953a, p. 221)

$$x^l e^{-\frac{1}{2}x}W_{k,m}(x)=G_{12}^{20}\left(x\,\middle|\,\begin{matrix}l-k+1\\m+l+\frac{1}{2},\,l-m+\frac{1}{2}\end{matrix}\right),\quad\ldots\quad\ldots\quad(26)$$

if we put $m=s=2$, $r=1$, $n=0$, $\alpha_1=l-k+1$, $\beta_1=l+m+\frac{1}{2}$ and $\beta_2=l-m+\frac{1}{2}$ in (23) we get

$$\int_0^\infty x^{l+\delta-1}e^{-\frac{1}{2}x}W_{k,m}(x)\,{}_2F_1(a,b;c;1-p/x)\,dx$$

$$=p^\delta\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)\Gamma(c-b)\Gamma(b)}G_{34}^{42}\left(p\,\middle|\,\begin{matrix}1-\delta-a,\,1-\delta-b,\,l-k+1\\-\delta,\,c-a-b-\delta,\,m+l+\frac{1}{2},\,l-m+\frac{1}{2}\end{matrix}\right),\quad(27)$$

provided that $R(l\pm m+a+\delta+\frac{1}{2})>0$, $R(l\pm m+b+\delta+\frac{1}{2})>0$, $R(c-a-b)>0$ and $R(p)>0$.

(c) If we take $m=s=2$, $n=r=1$, $\alpha_1=k+l+1$, $\beta_1=l-m+\frac{1}{2}$, $\beta_2=l+m+\frac{1}{2}$ and apply the relation (Erdelyi, 1953a, p. 221)

$$G_{12}^{21}\left(x\,\middle|\,\begin{matrix}l+k+1\\l-m+\frac{1}{2},\,l+m+\frac{1}{2}\end{matrix}\right)=\Gamma(\tfrac{1}{2}+m-k)\Gamma(\tfrac{1}{2}-m-k)x^l e^{x/2}W_{k,m}(x)\quad(28)$$

the result (23) reduces to

$$\int_0^\infty x^{l+\delta-1}e^{\frac{1}{2}x}W_{k,m}(x){}_2F_1(a,b;c;1-p/x)\,dx$$

$$=p^\delta\frac{\Gamma(c)}{\Gamma(\frac{1}{2}+m-k)\Gamma(\frac{1}{2}-m-k)\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}$$

$$G_{34}^{43}\left(p\,\middle|\,\begin{matrix}1-\delta-a,\,1-\delta-b,\,k+l+1\\-\delta,\,c-a-b-\delta,\,l-m+\frac{1}{2},\,l+m+\frac{1}{2}\end{matrix}\right),\quad\ldots\quad(29)$$

provided that $R(l+\delta+k)>0$, $R(l\pm m+\delta+a+\frac{1}{2})>0$, $R(l\pm m+\delta+b+\frac{1}{2})>0$, $R(p)>0$ and $R(c-a-b)>0$.

(d) Taking $m = s = 4$, $n = 0$, $r = 2$, $\alpha_1 = l+k+1$, $\alpha_2 = l-k+1$, $\beta_1 = l+\frac{1}{2}$, $\beta_2 = l+1$, $\beta_3 = l+m+\frac{1}{2}$, $\beta_4 = l-m+\frac{1}{2}$ in (23) and then making use of the relation (Erdelyi, 1953*a*, p. 222)

$$G_{24}^{40}\left(\frac{x^2}{4}\middle| \begin{matrix} \frac{1}{2}l+k+1,\ \frac{1}{2}l-k+1 \\ \frac{1}{2}l+\frac{1}{2},\ \frac{1}{2}l+1,\ \frac{1}{2}l+m+\frac{1}{2},\ \frac{1}{2}l-m+\frac{1}{2} \end{matrix}\right) = \pi^{\frac{1}{2}} x^l W_{k,\,m}(x) W_{-k,\,m}(x) \tag{30}$$

we have

$$\int_0^\infty x^{l+\delta-1} W_{k,\,m}(2\sqrt{x}) W_{-k,\,m}(2\sqrt{x})\, {}_2F_1(a, b\,; c\,; 1-p/x)\, dx$$

$$= p^\delta \pi^{-\frac{1}{2}} 2^{-2l} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} \times$$

$$G_{46}^{62}\left(p\middle| \begin{matrix} 1-\delta-a,\ 1-\delta-b,\ l+k+1,\ l-k+1 \\ -\delta,\ c-a-b-\delta,\ l+\frac{1}{2},\ l+1,\ l+m+\frac{1}{2},\ l-m+\frac{1}{2} \end{matrix}\right)$$

provided that $R(l+\delta\pm m+a+\frac{1}{2}) > 0$, $R(l+\delta\pm m+b+\frac{1}{2}) > 0$, $R(c-a-b) > 0$ and $R(p) > 0$.

Acknowledgement

The author is highly grateful to Dr. C. B. Rathie for his guidance and keen interest in the preparation of this paper.

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