DETERMINATION OF THE JUMP OF A FUNCTION BY ITS FOURIER SERIES

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1.1. Suppose that $f(\theta)$ is integrable (L) over $(-\pi, \pi)$ and periodic outside this range with period 2π. Let

$$f(\theta) \sim \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos n\theta+b_n \sin n\theta). \qquad \qquad (1.1.1)$$

Then the series conjugate to the above Fourier series is

$$\sum_{n=1}^{\infty}(b_n \cos n\theta-a_n \sin n\theta)=\sum_{n=1}^{\infty} B_n(\theta). \qquad \qquad (1.1.2)$$

DEFINITION (Riesz, 1909). A series Σa_n is said to be summable (R, w, k), $k > -1$, to sum s, if $C_k(w) \to s$, as $w \to \infty$, where

$$C_k(w)=w^{-k} \sum_{n \leqslant w}(w-n)^k a_n$$

denotes the (R, w, k) mean of Σa_n.

We make use of the following notations

$$\gamma_\alpha(x)=\int_0^1(1-t)^{\alpha-1} \cos xt \, dt, \ \alpha>0 ;$$

$$\bar{\gamma}_\alpha(x)=\int_0^1(1-t)^{\alpha-1} \sin xt \, dt, \ \alpha>0 ;$$

$$\psi(t)=\tfrac{1}{2}\{f(x+t)-f(x-t)\} ;$$

$$\theta(t)=\psi(t)-l ;$$

$$\Psi_0(t)=\psi(t) ;$$

$$\Psi_\alpha(t)=\frac{1}{\Gamma(\alpha)} \int_0^t(t-u)^{\alpha-1} \psi(u) \, du, \ \alpha>0 ;$$

$$\psi_\alpha(t)=\Gamma(\alpha+1) t^{-\alpha} \Psi_\alpha(t), \ \alpha \geqslant 0 ;$$

and define $\Theta_\alpha(t)$ and $\theta_\alpha(t)$ in a similar way. Also we denote by $\tau_\beta(w)$ and $C_\beta(w)$ the (R, w, β) means of the sequence $\{nB_n(x)\}$ and the series $\Sigma B_n(x)$, respectively.

If $\theta_\alpha(t)$, defined as above, tends to zero, as $t \to 0$, then the function $f(\theta)$ is said to have the generalized jump l, at $\theta = x$, in mean of order α.

The object of the present paper is to determine the generalized jump of $f(\theta)$, at $\theta = x$, by considering the summability of the sequence $\{nB_n(x)\}$.

1.2. Fejér (1913) proved for the first time that if $\theta(t) \to 0$, as $t \to 0$, then the sequence $\{nB_n(x)\}$ is summable (C, k), $k > 1$, to sum $2l/\pi$. For any $\alpha \geqslant 0$, it was stated by Chow (1942) and also proved by Minakshisundaram (1944) that if $\theta_\alpha(t) = o(1)$, as $t \to 0$, then the sequence $\{nB_n(x)\}$ is summable $(C, \alpha+1+\delta)$, to sum $2l/\pi$, for $\delta > 0$.

To obtain the above result for $\delta = 0$, Mohanty and Nanda (1954) have proved, for the case $\alpha = 0$, that if

$$\text{(i)}\ \theta(t) = o\left(1/\log\frac{1}{t}\right),\ \text{as } t \to 0, \text{ and (ii) } a_n,\ b_n = O(n^{-\delta}),\ \delta > 0,$$

then the sequence $\{nB_n(x)\}$ is summable $(C, 1)$, to sum $2l/\pi$. The particular case $\rho = 0$ of Theorem 1 of this paper shows that if the condition (i) is replaced by

$$\text{(i)}'\ \theta_\alpha(t) = o\left(1/\log\frac{1}{t}\right),\ \text{as } t \to 0,\ \alpha > 0$$

then the sequence $\{nB_n(x)\} \to \dfrac{2l}{\pi}(C, \alpha+1)$, without any condition corresponding to (ii) being imposed.

In general, knowing the order of $\theta_\alpha(t)$, as $t \to 0$, for $\alpha > 0$, we have studied in this paper the behaviour of $\tau_\beta(w)$, as $w \to \infty$, for particular values of α and β and have investigated conditions which may yield the (R, w, β) summability of the sequence $\{nB_n(x)\}$.

Another method of determining the jump of $f(\theta)$, at $\theta = x$, was introduced by Szász in 1938. Instead of considering the summability of the sequence $\{nB_n(x)\}$, he dealt with the behaviour of the difference of two Cesàro means, of the same order of the conjugate series (1.1.2), at $\theta = x$. He proved that if

$$\text{(iii)}\ \int_0^t |\theta(u)|\,du = o(t),\ \text{as } t \to 0$$

then

$$\lim_{n\to\infty} \{C_1(2n) - C_1(n)\} = \frac{2l}{\pi}\log 2,$$

$C_1(n)$ denoting the nth $(C, 1)$ mean of (1.1.2), at $\theta = x$. This result has been further improved by Maruyama in 1939. Chow (1941) has obtained a more refined result by proving that if

$$\text{(iv)}\ \int_0^t |\theta(u)|\,du = O(t),\ \text{as } t \to 0,$$

and

$$\text{(v)}\ \int_0^t \theta(u)\,du = o(t),\ \text{as } t \to 0$$

then, for m/n tending to $d > 1$, as $n \to \infty$, $m \to \infty$,

$$\lim_{n\to\infty} [C_\delta(m) - C_\delta(n)] = \frac{2l}{\pi}\log d,$$

for every $\delta > 0$.

Replacing $\theta(u)$ in conditions (iv) and (v), by $\theta_\alpha(u)$, where $\alpha \geqslant 0$, Minakshisundaram (1944) has obtained the above result for the difference of $C_{\alpha+\delta}(m)$ and $C_{\alpha+\delta}(n)$, where $\delta > 0$.

In Theorem 4 of the present paper, we have obtained a sufficient condition on $\theta_\alpha(u)$, in order that the above result may hold good for $\delta = 0$.

In § 5 we have given some results, in the form of corollaries, for the behaviour of the Riesz means of the conjugate series (1.1.2), at $\theta = x$, which have been deduced directly from Theorems 1, 2 and 3 with the help of a Tauberian theorem due to Hardy and Littlewood (1931).

2.1. THEOREM 1. *If, for* $\alpha > 0$, $-1 < \rho < 1$, $\alpha+\rho > 0$,

$$\int_0^t |\theta_\alpha(u)|\,du = o\left(t^{\rho+1}/\log\frac{1}{t}\right), \text{ as } t \to 0,$$

then we have

$$\tau_{\alpha+\rho+1}(w) - \frac{2l}{\pi} = o(w^{-\rho}), \quad \text{as } w \to \infty.$$

THEOREM 2. *If, for* $\alpha > 0$, $-1 < \rho < 1$, $\alpha+\rho > 0$, $p > -1$,

$$\int_t^\pi |\theta_\alpha(u)|/u\,du = o\left\{t^\rho\left(\log\frac{1}{t}\right)^{p+1}\right\}, \quad \text{as } t \to 0,$$

then we have

$$\tau_{\alpha+\rho+1}(w) - \frac{2l}{\pi} = o\{w^{-\rho}(\log w)^{p+1}\}, \quad \text{as } w \to \infty.$$

2.2. We require the following lemmas for the proofs of Theorems 1 and 2.

LEMMA 1.

(a) $\gamma_k(0) = \frac{1}{k}$, $k > 0$ (Hobson, 1925, p. 565);

(b) $\bar{\gamma}_{k+1}(x) - \bar{\gamma}_k(x) = \gamma'_k(x)$, $k > 0$ (Bosanquet and Hyslop, 1937);

(c) *If h is a positive integer and* $\chi = \min(\alpha, h+2)$, *then*

$\gamma_\alpha^{(h)}(x) \leqslant A(1+x)^{-\chi}$, *A being a constant* (Bosanquet and Hyslop, 1937).

LEMMA 2. (*See* Bosanquet and Hyslop, 1937.) *If* $\tau_\alpha(w)$ *and* $C_\alpha(w)$ *denote the* (R, w, α) *means of the sequence* $\{nu_n\}$ *and the series* Σu_n, *respectively, then*

$$\tau_{\alpha+1}(w) = (\alpha+1)\{C_\alpha(w) - C_{\alpha+1}(w)\} = wC'_{\alpha+1}(w), \; \alpha > 0.$$

LEMMA 3. (Prasad, 1931.)

$$C_k(w) = \frac{2w}{\pi}\int_0^\infty \psi(t)\bar{\gamma}_{k+1}(wt)\,dt, \; k > 0.$$

LEMMA 4. *If* $[\alpha]$ *and* $\{\alpha\}$ *denote the integral and fractional parts of a non-integral number* $\alpha > 0$, *and if* $r < 1$, $-1 < \rho < 1$, $\alpha+\rho > 0$, $u > \frac{1}{w}$, *then*

$$J(w, u) = w^{[\alpha]+2} u^{\alpha+\rho+1} \int_u^{w^{-r}} (t-u)^{-\{\alpha\}} \gamma^{([\alpha]+2)}_{\alpha+\rho+1}(wt)\, dt$$
$$= O(w^{-\rho}).$$

Proof of Lemma 4. Putting

$$J(w, u) = w^{[\alpha]+2} u^{\alpha+\rho+1} \left\{ \int_u^{u+w^{-1}} + \int_{u+w^{-1}}^{w^{-r}} \right\} = J_1 + J_2, \text{ say,}$$

where the second integral does not exist if $u+w^{-1} > w^{-r}$, we have, from Lemma 1,

$$J_1 = O\left[w^{[\alpha]+2} u^{\alpha+\rho+1} \int_u^{u+w^{-1}} (t-u)^{-\{\alpha\}} (1+wt)^{-\alpha-\rho-1} dt \right]$$
$$= O\left[w^{[\alpha]+2} u^{\alpha+\rho+1} w^{\{\alpha\}-1} / (wu)^{\alpha+\rho+1} \right]$$
$$= O(w^{-\rho}),$$

and, by the second mean value theorem,

$$J_2 = O\left[w^{[\alpha]+2} u^{\alpha+\rho+1} \left(\frac{1}{w}\right)^{-\{\alpha\}} \int_{u+w^{-1}}^{\eta} \gamma^{([\alpha]+2)}_{\alpha+\rho+1}(wt)\, dt \right],$$

$$u+w^{-1} < \eta < w^{-r},$$

$$= O\left[w^{[\alpha]+2} u^{\alpha+\rho+1} w^{\{\alpha\}-1} / (wu)^{\alpha+\rho+1} \right]$$
$$= O(w^{-\rho}),$$

which completes the proof.

2.3. *Proof of Theorem 1.* From Lemmas 1, 2 and 3 we have

$$\tau_{\alpha+\rho+1}(w) = (\alpha+\rho+1)\{C_{\alpha+\rho}(w) - C_{\alpha+\rho+1}(w)\}$$
$$= (\alpha+\rho+1)\frac{2w}{\pi} \int_0^\infty \psi(t) \{\bar{\gamma}_{\alpha+\rho+1}(wt) - \bar{\gamma}_{\alpha+\rho+2}(wt)\}\, dt$$
$$= -(\alpha+\rho+1)\frac{2w}{\pi} \int_0^\infty \psi(t) \gamma'_{\alpha+\rho+1}(wt)\, dt$$
$$= -(\alpha+\rho+1)\frac{2w}{\pi} \left(\int_0^{w^{-r}} + \int_{w^{-r}}^\infty \right), \quad r < 1,$$
$$= I_1(w) + I_2(w), \text{ say,} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.3.1)$$

where $r < (\delta-1-\rho)/\delta$, $\delta = \min(\alpha+\rho+1, 3)$. Now

$$I_2(w) = -(\alpha+\rho+1)\frac{2w}{\pi}\int_{w^{-r}}^{\pi} \psi(t)\gamma'_{\alpha+\rho+1}(wt)\,dt$$

$$-(\alpha+\rho+1)\frac{2w}{\pi}\sum_{p=1}^{\infty}\left\{\int_{p\pi}^{(p+1)\pi} \psi(t)\gamma'_{\alpha+\rho+1}(wt)\,dt\right\}$$

$$= O\left\{w\int_{w^{-r}}^{\pi} |\psi(t)|/(wt)^{\delta}\,dt\right\}+\sum_{p=1}^{\infty} O\left\{w\int_{p\pi}^{(p+1)\pi} |\psi(t)|/(wt)^{\delta}\,dt\right\}$$

by Lemma 1(c),

$$= O\left\{\frac{w^{-\rho}}{w^{\delta-1-\rho-r\delta}}\right\}+O\left\{\frac{w^{-\rho}}{w^{\delta-1-\rho}}\right\}$$

$$= o(w^{-\rho}). \qquad \ldots \ldots \ldots \ldots \ldots \ldots \qquad (2.3.2)$$

Next, by Lemma 1(*a*), we have

$$I_1(w) = -\frac{2w(\alpha+\rho+1)}{\pi}\int_0^{w^{-r}} \theta(t)\gamma'_{\alpha+\rho+1}(wt)\,dt-\frac{2l}{\pi}(\alpha+\rho+1)$$

$$\times\int_0^{w^{-r}} \gamma'_{\alpha+\rho+1}(wt)\,dt$$

$$= -\frac{2w(\alpha+\rho+1)}{\pi}\int_0^{w^{-r}} \theta(t)\gamma'_{\alpha+\rho+1}(wt)\,dt+\frac{2l}{\pi}+O\{(wt)^{-q}\}_{t=w^{-r}},$$

where $q = \min(2, \alpha+\rho+1)$. Now choosing $r < (q-\rho)/q$, we have

$$I_1(w) = \frac{2l}{\pi}-\frac{2w(\alpha+\rho+1)}{\pi}\int_0^{w^{-r}} \theta(t)\gamma'_{\alpha+\rho+1}(wt)\,dt+o(w^{-\rho}). \quad \ldots \quad (2.3.3)$$

Thus it follows from (2.3.1) . . . . (2.3.3), and by integration by parts, that

$$\tau_{\alpha+\rho+1}(w)-\frac{2l}{\pi} = \frac{-2w(\alpha+\rho+1)}{\pi}\int_0^{w^{-r}} \theta(t)\gamma'_{\alpha+\rho+1}(wt)dt+o(w^{-\rho})$$

$$= \frac{2(\alpha+\rho+1)}{\pi}\left[\sum_{\nu=1}^{k+1}(-1)^{\nu}\Theta_{\nu}(t)\gamma^{(\nu)}_{\alpha+\rho+1}(wt)\,.\,w^{\nu}\right]_{t=w^{-r}}$$

$$+\frac{2(\alpha+\rho+1)}{\pi}(-1)^{k+2}w^{k+2}\int_0^{w^{-r}} \Theta_{k+1}(t)\gamma^{(k+2)}_{\alpha+\rho+1}(wt)\,dt+o(w^{-\rho})$$

$$= P_1+P_2+o(w^{-\rho}), \text{ say}, \qquad \ldots \ldots \ldots \ldots \qquad (2.3.4)$$

where $h = [\alpha]$. Now

$$P_1 = \sum_{\nu=1}^{h-2} O\{w^{\nu}t^{\nu-1}/(wt)^{\nu+2}\}_{t=w^{-r}} + O\{t^{h-2}w^{h-1}/(wt)^{p}\}_{t=w^{-r}}$$

$$+O\{t^{h-1}w^{h}/(wt)^{s}\}_{t=w^{-r}} + O\{w^{-\rho}w^{h+1}t^{\alpha+\rho}/(w^{\alpha+1}t^{\alpha+\rho+1})\}_{t=w^{-r}},$$

where $p = \min(h+1, \alpha+\rho+1)$, $s = \min(h+2, \alpha+\rho+1)$. Taking $r < \min((2-\rho)/3, (p+1-h-\rho)/(p+2-h), (s-h-\rho)/(s+1-h), \{\alpha\})$, we obtain

$$P_1 = o(w^{-\rho}). \qquad \qquad (2.3.5)$$

Next

$$P_2 = O\left[w^{h+2}\int_{1/w}^{w^{-r}} \gamma^{(h+2)}_{\alpha+\rho+1}(wt)\left\{\int_0^t (t-u)^{-\{\alpha\}}\Theta_{\alpha}(u)\,du\right\}dt\right]$$

$$+O\left[w^{h+2}\int_0^{w^{-1}} |\Theta_{h+1}(t)|\,dt\right]$$

$$= O\left[\int_{1/w}^{w^{-r}} \Theta_{\alpha}(u)\,.\,u^{-(\alpha+\rho+1)}J(w,u)\,du\right]+o(w^{-\rho}),$$

where

$$J(w,u) = w^{h+2}u^{\alpha+\rho+1}\int_u^{w^{-r}} (t-u)^{-\{\alpha\}}\gamma^{(h+2)}_{\alpha+\rho+1}(wt)\,dt = O(w^{-\rho}),$$

by Lemma 4. Hence we have, writing $\theta^*_{\alpha}(t) = \int_0^t |\theta_{\alpha}(u)|\,du$,

$$P_2 = o(w^{-\rho})+O\left\{w^{-\rho}\int_{1/w}^{w^{-r}} |\theta_{\alpha}(u)|/u^{\rho+1}\,du\right\}$$

$$= o(w^{-\rho})+O\left\{w^{-\rho}\left(\theta^*_{\alpha}(u)/u^{\rho+1}\right)_{u=w^{-1}}^{w^{-r}}\right\}+O\left\{w^{-\rho}\int_{1/w}^{w^{-r}} \theta^*_{\alpha}(u)/u^{2+\rho}\,du\right\}$$

$$= o(w^{-\rho})+o\left\{w^{-\rho}\int_{1/w}^{w^{-r}} 1/(u\log u)\,du\right\}$$

$$= o(w^{-\rho}). \qquad \qquad (2.3.6)$$

The results (2.3.4) .... (2.3.6) yield

$$\tau_{\alpha+\rho+1}(w) - \frac{2l}{\pi} = o(w^{-\rho}),$$

which completes the proof for non-integral values of $\alpha > 0$. The proof for integral values of $\alpha \geqslant 1$ follows easily.

2.4. *Proof of Theorem 2.* Proceeding exactly as in the proof of Theorem 1 and writing $\eta(t)=\int_t^{\pi} |\theta_\alpha(u)|/u\,du$, we have

$$\tau_{\alpha+\rho+1}(w)-\frac{2l}{\pi}=o\{w^{-\rho}(\log w)^{\rho+1}\}+O\left\{w^{-\rho}\int_{1/w}^{w^{-r}}|\theta_\alpha(u)|/u^{\rho+1}\,du\right\}$$

$$=o\{w^{-\rho}(\log w)^{\rho+1}\}+O\{w^{-\rho}\eta(t)t^{-\rho}\}_{t=w^{-1}}^{w^{-r}}+O\left\{w^{-\rho}\int_{w^{-1}}^{w^{-r}}\frac{\eta(u)}{u^{\rho+1}}\,du\right\}$$

$$=o\{w^{-\rho}(\log w)^{\rho+1}\},$$

which gives the required result.

3.1. THEOREM 3(A). *If, for* $1>\rho>0,\ 0<\beta\leqslant 2(\rho+1)$,

$$\int_0^t |\theta_\beta(u)|\,du=o(t^{\rho+1}),\quad \text{as } t\to 0,$$

then we have

$$\tau_{\alpha+1}(w)-\frac{2l}{\pi}=o(1),\ \text{as } w\to\infty,\ \text{for } \alpha\geqslant\beta/(\rho+1).$$

(B). *If* $\beta>2(\rho+1),\ 1>\rho>0$, *and* $\{\beta\}$ *denotes the fractional part of* β *and*

$$\int_0^t |\theta_\beta(u)|\,du=o(t^{\rho+1}),\ \text{as } t\to 0,$$

then we have

$$\tau_{\alpha+1}(w)-\frac{2l}{\pi}=o(1),\ \text{as } w\to\infty,\ \text{for } \alpha\geqslant\beta-(1+\{\beta\})\rho/(\rho+1).$$

3.2. Following lemmas will be used in the proof of Theorem 3.

LEMMA 5. (Riesz, 1922.) *If $V(x)$ and $W(x)$ are non-decreasing functions of x, for $x>0$, and if*

$$(\alpha)\ \xi(x)=o\{V(x)\},\ \text{and}\ (\beta)\ \xi_\alpha(x)=o\{W(x)\},$$

where

$$\xi_\alpha(x)=\alpha\int_0^x (x-u)^{\alpha-1}\xi(x)\,dx,\ \alpha>0,$$

then

$$\xi_q(x)=o[\{V(x)\}^{1-q/\alpha}\{W(x)\}^{q/\alpha}],\ 0<q<\alpha.$$

LEMMA 6. *If β is a non-integral number >0, $\{\beta\}$ denoting its fractional part, and k is an integer such that $k-1<\beta<k$, then we have, for $u>w^{-1}$ and a constant c,*

$$J(w,u)=w^{k+1}\int_u^{cw^{-r}}\gamma_{\alpha+1}^{k+1}(wt)(t-u)^{-\{\beta\}}\,du,\ r<1,\ \alpha<\beta,$$

$$=O(w^{\beta-\alpha}/u^{\alpha+1}).$$

The proof of the above lemma is similar to that of Lemma 4.

3.3. *Proof of Theorem 3(A).* Proceeding as in the proof of Theorem 1 and taking c to be sufficiently large number, we have

$$\tau_{\alpha+1}(w) = -\frac{2w}{\pi}(\alpha+1)\int_0^\infty \psi(t)\gamma'_{\alpha+1}(wt)\,dt$$

$$= -(\alpha+1)\frac{2w}{\pi}\left\{\int_0^{cw^{-r}} + \int_{cw^{-r}}^\infty\right\},\quad r=\alpha/(1+\alpha),$$

$$= Q_1+Q_2, \text{ say.}$$

Supposing that $\alpha \leqslant 2$, it follows from Lemma 1(c), that

$$Q_2 = O\left\{w\int_{cw^{-r}}^{\pi} |\psi(t)|/(wt)^{\alpha+1}\,dt\right\} + O\left\{w^{-\alpha}\sum_{p=1}^{\infty}\int_{p\pi}^{(p+1)\pi} |\psi(t)|/t^{\alpha+1}\,dt\right\}$$

$$= O(c^{-\alpha-1}) = o(1),$$

as $c \to \infty$ after $w \to \infty$.

Next we have if k is an integer such that for non-integral values of β, $k-1<\beta<k$, then

$$Q_1 - \frac{2l}{\pi} = -\frac{2w(\alpha+1)}{\pi}\int_0^{cw^{-r}} \theta(t)\gamma'_{\alpha+1}(wt)\,dt + o(1)$$

$$= -\frac{2(\alpha+1)}{\pi}\left\{\sum_{\nu=1}^{k}(-1)^{\nu-1}\Theta_\nu(t)\gamma^{(\nu)}_{\alpha+1}(wt)\,.\,w^\nu\right\}_{t=cw^{-r}}$$

$$+\frac{2(\alpha+1)}{\pi}(-1)^{k+1}w^{k+1}\int_0^{cw^{-r}}\Theta_k(t)\gamma^{(k+1)}_{\alpha+1}(wt)\,dt + o(1)$$

$$= Q_{1,1}+Q_{1,2}+o(1), \text{ say.}$$

By Lemma 5,

$$Q_{1,1} = o\{w/(wt)^{\alpha+1}\}_{t=cw^{-r}} + \sum_{\nu=2}^{k} o\{w^\nu t^{(\nu-1)(\beta+\rho+1)/\beta}/(wt)^{\alpha+1}\}_{t=cw^{-r}}$$

$$= o(1) + o\left\{\sum_{\nu=2}^{k} 1/w^{\alpha+1-\nu-r\{(\alpha+1)\beta-(\nu-1)(1+\beta+\rho)\}/\beta}\right\} = o(1),$$

provided that

$$\beta(\alpha+1-\nu)(\alpha+1) \geqslant \alpha\{(\alpha+1)\beta-(\nu-1)(\beta+\rho+1)\}$$

or

$$\beta < \alpha(\rho+1),$$

which is seen to be true by hypothesis. Next

$$Q_{1,2} = o(1) + O\left\{\int_{1/w}^{cw^{-r}} |\Theta_\beta(u)|\,J(w,u)\,du\right\} + O\left[w^{k+1}\int_0^{w^{-1}}|\Theta_k(t)|\,dt\right],$$

where

$$J(w, u) = w^{k+1} \int_u^{cw^{-r}} \gamma_{\alpha+1}^{(k+1)}(wt)(t-u)^{-\{\beta\}} du$$
$$= O(w^{\beta-\alpha}/u^{\alpha+1}),$$

by Lemma 6. Hence

$$Q_{1,2} = o(1) + O\left\{ w^{\beta-\alpha} \int_{1/w}^{cw^{-r}} |\Theta_\beta(u)| / u^{\alpha+1} du \right\}$$
$$= o(1) + O\left[w^{\beta-\alpha} \left\{ \int_0^u |\Theta_\beta(u)|\, du \,.\, u^{-\alpha-1} \right\}_{u=1/w}^{cw^{-r}} \right]$$
$$+ O\left[w^{\beta-\alpha} \int_{1/w}^{cw^{-r}} u^{-\alpha-2} \left\{ \int_0^u |\Theta_\beta(u)|\, dt \right\} du \right]$$
$$= o(1) + o\left(w^{\beta-\alpha-\alpha(\beta-\alpha+\rho)/(\alpha+1)}\right)$$
$$= o(1),$$

provided that $\alpha \geqslant \beta/(\rho+1)$. This completes the proof of Theorem 3(A) for non-integral values of $\beta \leqslant 2(\rho+1)$. For integral values of β the proof follows easily.

Proof of Theorem 3(B). Suppose first that β is non-integral and > 2. Writing $h = [\beta]-1$, $k = [\beta]+1 = h+2$, and assuming $\alpha > \beta - 1$, we have

$$\tau_{\alpha+1}(w) - \frac{2l}{\pi} = -\frac{2(\alpha+1)w}{\pi} \int_0^\infty \theta(t) \gamma'_{\alpha+1}(wt) dt$$
$$= -\frac{2(\alpha+1)}{\pi} \sum_{\nu=1}^{h} \left[(-1)^{\nu-1} w^\nu \Theta_\nu(t) \gamma_{\alpha+1}^{(\nu)}(wt) \right]_{t=0}^{\infty}$$
$$+ (-1)^{h+1} w^{h+1} \int_0^\infty \Theta_h(t) \gamma_{\alpha+1}^{(h+1)}(wt) dt$$
$$= (-1)^{h+1} w^{h+1} \int_0^\infty \Theta_h(t) \gamma_{\alpha+1}^{(h+1)}(wt) dt$$
$$= (-1)^{h+1} w^{h+1} \int_0^{cw^{-r}} \Theta_h(t) \gamma_{\alpha+1}^{(h+1)}(wt) dt$$
$$+ O\left[w^{h+1} \int_{cw^{-r}}^{\pi} |\theta_h(t)| / (w^{\alpha+1} t^{\alpha+1-h}) dt \right] +$$
$$+ O\left[w^{h-\alpha} \sum_{p=1}^{\infty} \int_{p\pi}^{(p+1)\pi} |\theta_h(t)| / t^{\alpha+1-h} dt \right]$$
$$= (-1)^{h+1} w^{h+1} \int_0^{cw^{-r}} \Theta_h(t) \gamma_{\alpha+1}^{(h+1)}(wt) dt + o(1),$$

where $r = (\alpha-h)/(\alpha+1-h)$, and $c \to \infty$ after $w \to \infty$. Now integrating twice by parts and proceeding as before, we get

$$\tau_{\alpha+1}(w) - \frac{2l}{\pi} = o(1) + O\left\{ w^{h+1}\Theta_{h+1}(t)/(wt)^{\alpha+1} \right\}_{t=cw^{-r}} + $$

$$+ O\left\{ w^{h+2}\Theta_{h+2}(t)/(wt)^{\alpha+1} \right\}_{t=cw^{-r}}$$

$$+ O\left\{ w^{h+3} \int_0^{w^{-1}} |\Theta_k(t)|\, dt \right\} + (-1)^{k+1} w^{k+1} \int_{w^{-1}}^{cw^{-r}} \gamma_{\alpha+1}^{(k+1)}(wt) \, . \, \Theta_k(t)\, dt$$

$$= o(1) + O\left\{ \int_{w^{-1}}^{cw^{-r}} \Theta_\beta(u) \, . \, J(w, u)\, du \right\},$$

where

$$J(w, u) = w^{k+1} \int_u^{cw^{-r}} (t-u)^{-\{\beta\}} \gamma_{\alpha+1}^{(k+1)}(wt)\, dt = O\{ w^{\beta-\alpha} u^{-\alpha-1} \},$$

by Lemma 6. Hence we have, on writing

$$\theta_\beta^*(t) = \int_0^t |\theta_\beta(u)|\, du,$$

$$\tau_{\alpha+1}(w) - \frac{2l}{\pi} = o(1) + O\left\{ \int_{w^{-1}}^{cw^{-r}} |\theta_\beta(u)|\, u^{-\alpha-1+\beta} w^{\beta-\alpha}\, du \right\}$$

$$= o(1) + O\left[w^{\beta-\alpha} \left\{ \theta_\beta^*(u)/u^{\alpha+1-\beta} \right\}_{u=w^{-1}}^{cw^{-r}} \right] +$$

$$+ O\left[\int_{w^{-1}}^{cw^{-r}} \theta_\beta^*(u) \, . \, w^{\beta-\alpha} \, . \, u^{-\alpha-2+\beta}\, du \right]$$

$$= o(1) + o\left[w^{\beta-\alpha-r(\beta+\rho-\alpha)} \right], \quad r = (\alpha-h)/(\alpha+1-h),$$

$$= o(1),$$

provided that

$$\alpha \geqslant \beta - (1+\{\beta\})\rho/(1+\rho).$$

This completes the proof of Theorem 3(B) for non-integral values of β. For the case when β is an integer, $\{\beta\} = 0$, and the analysis gets simpler.

4. THEOREM 4. *If, for* $\alpha > 0$,

$$\int_0^t |\theta_\alpha(u)|\, du = o\left(t/\log \frac{1}{t} \right), \text{ as } t \to 0, \quad \ldots \quad \ldots \quad \ldots \quad (4{\cdot}1)$$

then we have

$$\lim_{w\to\infty} \{ C_\alpha(\lambda w) - C_\alpha(w) \} = \frac{2l}{\pi} \log \lambda, \; \lambda > 1.$$

Proof of Theorem 4. From Lemma 2

$$C'_{\alpha+1}(w) = \frac{1}{w}\tau_{\alpha+1}(w).$$

Hence it follows from Theorem 1, that

$$C_{\alpha+1}(\lambda w) - C_{\alpha+1}(w) = \int_w^{\lambda w} C'_{\alpha+1}(x)\,dx = \int_w^{\lambda w} \tau_{\alpha+1}(x)/x\,dx$$

$$\sim \frac{2l}{\pi}\int_w^{\lambda w}\frac{dx}{x} + o\left\{\int_w^{\lambda w}\frac{dx}{x}\right\}$$

$$\sim \frac{2l}{\pi}\log\lambda.$$

Also

$$C_\alpha(\lambda w) - C_\alpha(w) = C_{\alpha+1}(\lambda w) - C_{\alpha+1}(w) + \frac{1}{\alpha+1}\{\tau_{\alpha+1}(\lambda w) - \tau_{\alpha+1}(w)\}$$

$$= \frac{2l}{\pi}\log\lambda + o(1),$$

by (4.1), which completes the proof.

5. Following lemma is due to Hardy and Littlewood (1931).

LEMMA 7. *If Σa_n is summable (A), then a necessary and sufficient condition that it should be summable (C, k), $k > -1$, is that the sequence $\{nu_n\}$ is summable $(C, k+1)$ to the value zero.*

With the help of Theorems 1, 2 and 3 and Lemmas 2 and 7, we obtain the following results:

COROLLARY 1. *If, for $\alpha > 0$, $-1 < \rho < 1$, $\alpha+\rho > 0$,*

$$\text{(i)}\quad \int_0^t |\psi_\alpha(u)|\,du = o\left(t^{\rho+1}\Big/\log\frac{1}{t}\right),\ as\ t\to 0,$$

and further if for the range $\rho \geqslant 0$,

$$\text{(ii)}\quad \lim_{t\to 0}\frac{2}{\pi}\int_t^\infty \psi(u)/u\,du \to s(c, k),$$

for some value of $k > 0$, then we have

$$C_{\alpha+\rho}(w) - s = o(w^{-\rho}),\ as\ w\to\infty.$$

COROLLARY 2(A). *If, for $\alpha > 0$, $-1 < \rho < 0$, $\alpha+\rho > 0$, $p > -1$,*

$$\int_t^\pi |\psi_\alpha(u)|/u\,du = o\left\{t^\rho\left(\log\frac{1}{t}\right)^{p+1}\right\},\ as\ t\to o,$$

then we have

$$C_{\alpha+\rho}(w) = o\{w^{-\rho}(\log w)^{p+1}\},\ as\ t\to 0.$$

(B). *If, for* $\alpha > 0$, $\rho > 0$, $-1 < p < \infty$,

$$\text{(i)} \int_0^t |\psi_\alpha(u)|\,du = o\left\{t^\rho\left(\log\frac{1}{t}\right)^p\right\},\ as\ t \to 0,$$

and

$$\text{(ii)} \lim_{t\to 0}\frac{2}{\pi}\int_t^\infty \psi(u)/u\,du \to s(c,k)$$

for some value of $k > 0$, *then we have*

$$C_{\alpha+\rho}(w)-s = o\{w^{-\rho}(\log w)^{p+1}\},\ as\ w \to \infty.$$

Corollary 3(A). *If, for* $0 < \beta \leqslant 2(\rho+1)$, $1 > \rho > 0$,

$$\int_0^t |\psi_\beta(u)|\,du = o(t^{\rho+1}),\ as\ t \to o$$

and

$$\lim_{t\to 0}\frac{2}{\pi}\int_t^\infty \psi(u)/u\,du \to s(c,k),$$

for some value of $k > 0$, *then*

$$C_\alpha(w)-s = o(1),\ as\ w \to \infty,\ for\ \alpha > \beta/(\rho+1).$$

(B). *Let* $\beta > 2(\rho+1)$, $1 > \rho > 0$, *and* $\{\beta\}$ *denote the fractional part of* β. *If*

$$\text{(i)} \int_0^t |\psi_\beta(u)|\,du = o(t^{\rho+1}),\ as\ t \to o,$$

and

$$\text{(ii)} \lim_{t\to 0}\frac{2}{\pi}\int_t^\infty \psi(u)/u\,du \to s(c,k),$$

for some value of $k > 0$, *then we have*

$$C_\alpha(w)-s = o(1),\ as\ w \to \infty,\ for\ \alpha \geqslant \beta-(1+\{\beta\})\rho/(1+\rho).$$

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