ELASTO-DYNAMIC PROBLEM CONCERNING A CENTRE OF ROTATION IN A SEMI-INFINITE MEDIUM OF TRANSVERSELY ISOTROPIC MATERIAL

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ABSTRACT

The form of a nucleus of disturbance generating diverging waves of the rotatory type in a solid of transversely isotropic material has been directly found out. The problem of disturbances due to such a nucleus in a semi-infinite medium has been solved and the displacement on the surface has been obtained.

INTRODUCTION

The effects of various nuclei of strain depending on time and situated in an isotropic elastic medium were discussed in detail by Love (1904). The important cases which he considered are those of a centre of compression representing irrotational waves and a centre of rotation representing equivoluminal waves. Lamb (1904), Nakano (1925), Lapwood (1949), and others have considered the effect of various time-dependent centres of compression in a semi-infinite isotropic solid.

The possibility of the existence of similar nuclei in transversely isotropic medium and their effects have not yet been studied. In this paper, we have directly obtained the form of a possible centre of rotation depending on time in a transversely isotropic medium. Also the effects of such a centre of rotation operating at a depth below the free surface of such a semi-infinite solid have been found.

1. *Equations of motion:* The equations of motion in cylindrical co-ordinates in the absence of body forces are

$$\frac{\partial \widehat{rr}}{\partial r}+\frac{1}{r}\frac{\partial \widehat{r\theta}}{\partial \theta}+\frac{\partial \widehat{rz}}{\partial z}+\frac{\widehat{rr}-\widehat{\theta\theta}}{r}=\rho\frac{\partial^2 u_r}{\partial t^2} \qquad (1)$$

$$\frac{\partial \widehat{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \widehat{\theta\theta}}{\partial \theta}+\frac{\partial \widehat{\theta z}}{\partial z}+\frac{2\widehat{r\theta}}{r}=\rho\frac{\partial^2 u_\theta}{\partial t^2} \qquad (2)$$

$$\frac{\partial \widehat{rz}}{\partial r}+\frac{1}{r}\frac{\partial \widehat{\theta z}}{\partial \theta}+\frac{\partial \widehat{zz}}{\partial z}+\frac{\widehat{rz}}{r}=\rho\frac{\partial^2 u_z}{\partial t^2} \qquad (3)$$

where u_r, u_θ, u_z are the displacement components in cylindrical co-ordinates and ρ the density of the medium. The stress strain relations in a transversely isotropic medium are (Love, 1927, Art. 110)

$$\left.\begin{aligned}\widehat{rr}&=c_{11}e_{rr}+c_{12}e_{\theta\theta}+c_{13}e_{zz}; & \widehat{rz}&=c_{44}e_{rz}\\ \widehat{\theta\theta}&=c_{12}e_{rr}+c_{11}e_{\theta\theta}+c_{13}e_{zz}; & \widehat{r\theta}&=\tfrac{1}{2}(c_{11}-c_{12})e_{r\theta}\\ \widehat{zz}&=c_{13}e_{rr}+c_{13}e_{\theta\theta}+c_{33}e_{zz}; & \widehat{\theta z}&=c_{44}e_{\theta z}\end{aligned}\right\} \qquad (4)$$

where c_{11}, c_{12}, c_{13}, c_{33}, c_{44} are the elastic constants.

For a centre of rotation we take $u_r = u_z = 0$ and u_θ to be independent of θ. Then u_θ satisfies the equation of motion,

$$\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}+\frac{2c_{44}}{c_{11}-c_{12}}\frac{\partial^2 u_\theta}{\partial z^2}=\frac{2\rho}{c_{11}-c_{12}}\frac{\partial^2 u_\theta}{\partial t^2} \quad \ldots \quad \ldots \quad (5)$$

If we write

$$k^2=\frac{2c_{44}}{c_{11}-c_{12}}, \quad z_1^2=\frac{z^2}{k^2}, \quad \frac{c_{11}-c_{12}}{\rho}=\beta^2 \quad \ldots \quad \ldots \quad (6)$$

the equation (5) becomes in (r, θ, z_1) co-ordinates

$$\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}+\frac{\partial^2 u_\theta}{\partial z_1^2}=\frac{1}{\beta^2}\frac{\partial^2 u_\theta}{\partial t^2} \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

If we put $u_\theta = \dfrac{\partial \Phi}{\partial r}$ where Φ is a function of r, z and t, the equation (7) becomes

$$\frac{\partial}{\partial r}\left[\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{\partial^2\Phi}{\partial z_1^2}\right]=\frac{\partial}{\partial r}\left[\frac{1}{\beta^2}\frac{\partial^2\Phi}{\partial t^2}\right] \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

Equation (8) is satisfied if Φ satisfies the differential equation

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z_1^2}\right)\Phi=\frac{1}{\beta^2}\frac{\partial^2\Phi}{\partial t^2} \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

or

$$\nabla^2\Phi=\frac{1}{\beta^2}\frac{\partial^2\Phi}{\partial t^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9a)$$

In polar co-ordinates R, Θ, Ψ $(\equiv \theta)$ equation (9a) becomes

$$\frac{\partial^2\Phi}{\partial R^2}+\frac{2}{R}\frac{\partial\Phi}{\partial R}+\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial\Phi}{\partial\Theta}\right)=\frac{1}{\beta^2}\frac{\partial^2\Phi}{\partial t^2} \quad \ldots \quad \ldots \quad (9b)$$

where

$$R^2=r^2+z_1^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

2. *Form of the centre of rotation:* For a centre of rotation let us seek a solution of equation (9b) in which Φ is a function of R and t alone. Then Φ must satisfy the equation,

$$\nabla^2\Phi\equiv\frac{\partial^2\Phi}{\partial R^2}+\frac{2}{R}\frac{\partial\Phi}{\partial R}=\frac{1}{\beta^2}\frac{\partial^2\Phi}{\partial t^2} \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

Equation (10) can also be written as

$$\frac{\partial^2}{\partial R^2}(R\Phi)=\frac{1}{\beta^2}\frac{\partial^2}{\partial t^2}(R\Phi) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

Equation (12) has the general solution

$$\Phi=\frac{1}{R}F\left(t-\frac{R}{\beta}\right)+\frac{1}{R}f\left(t+\frac{R}{\beta}\right) \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

The first term represents a diverging wave and the second term which represents a reflected wave may be omitted since the medium is supposed to extend to infinity.

Therefore a nucleus of the form

$$\Phi = \frac{1}{R} F\left(t - \frac{R}{\beta}\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

is a possible one in a transversely isotropic medium. The form of Φ shows that diverging waves are being propagated from the nucleus with velocity β. Moreover, since only the azimuthal component u_θ is involved, it is a centre of rotation about the z-axis. The displacement at any point is

$$u_\theta = \frac{\partial \Phi}{\partial r}$$

$$= -\frac{r}{R}\left[\frac{1}{R^2} F\left(t - \frac{R}{\beta}\right) + \frac{1}{R} F'\left(t - \frac{R}{\beta}\right)\right] \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

The displacement at any point therefore consists of two terms of which one decays more rapidly than the other with distance from the nucleus.

The above results reduce to the case of isotropy if we use the following relations:

$$c_{11} = c_{33}\,;\; c_{13} = c_{12}\,;\; c_{44} = c_{66}\,:$$

$$c_{11} - 2c_{44} = c_{12}$$

3. *Disturbance produced by a centre of rotation in a semi-infinite solid:* Let the plane boundary of a semi-infinite solid of transversely isotropic material be given by $z = 0$, the axis of z being drawn into the solid.

Suppose in this solid a centre of rotation of the form

$$\Phi_1 = \frac{1}{R_1} \chi\left(t - \frac{R_1}{\beta}\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

acts at the point $(0, 0, c)$ where R_1 is given by

$$R_1^2 = r^2 + \frac{(z-c)^2}{k^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

(r, θ, z) being the co-ordinates of any point of the medium.

The boundary face is free from stresses. In our case, the r- and z-components of the displacement are identically zero. Therefore, the stresses $\widehat{zz}$, $\widehat{rz}$ across the plane $z = 0$ vanish identically. Hence the boundary condition to be satisfied is

$$(\widehat{\theta z})_{z=0} = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

In order to satisfy the boundary condition (18), we introduce an image of the nucleus at $(0, 0, -c)$ of the form

$$\Phi_2 = \frac{1}{R_2} \chi\left(t - \frac{R_2}{\beta}\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

where

$$R_2^2 = r^2 + \frac{(z+c)^2}{k^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

Then we consider the effect due to

$$\Phi = \Phi_1 + \Phi_2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

The resultant displacement is

$$u_\theta = \frac{\partial \Phi}{\partial r} = -\frac{r}{R_1^3}\chi\left(t-\frac{R_1}{\beta}\right) - \frac{r}{R_1^2\beta}\chi'\left(t-\frac{R_1}{\beta}\right)$$

$$-\frac{r}{R_2^3}\chi\left(t-\frac{R_2}{\beta}\right) - \frac{r}{R_2^3\beta}\chi'\left(t-\frac{R_2}{\beta}\right) \quad \ldots \quad \ldots \quad (22)$$

If

$$t_1 = t-\frac{R_1}{\beta}, \quad t_2 = t-\frac{R_2}{\beta}, \ldots \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

then

$$u_\theta = -\frac{r}{R_1^3}\chi(t_1) - \frac{r}{R_1^2\beta}\chi'(t_1) - \frac{r}{R_2^3}\chi(t_2) - \frac{r}{R_2^2\beta}\chi'(t_2) \quad \ldots \quad \ldots \quad (24)$$

$$\frac{\partial u_\theta}{\partial z} = \frac{3r(z-c)}{k^2R_1^5}\chi(t_1) + \frac{r(z-c)}{k^2R_1^4\beta}\chi'(t_1)$$

$$+\frac{r(z-c)}{k^2\beta^2R_1^3}\chi''(t_1) + \frac{2r(z-c)}{k^2\beta R_1^4}\chi'(t_1)$$

$$+\frac{3r(z+c)}{k^2R_2^5}\chi(t_2) + \frac{r(z+c)}{k^2R_2^4\beta}\chi'(t_2)$$

$$+\frac{r(z+c)}{k^2\beta^2R_2^3}\chi''(t_2) + \frac{2r(z+c)}{k^2\beta R_1^4}\chi'(t_1) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

Now the boundary condition (18) reduces to

$$\left(c_{44}\frac{\partial u_\theta}{\partial z}\right)_{z=0} = 0.$$

On putting $z = 0$ in (25) we find that the condition

$$\left(\frac{\partial u_\theta}{\partial z}\right)_{z=0} = 0 \text{ is satisfied.}$$

Thus the boundary condition (18) is satisfied for the displacement potential $\Phi = \Phi_1+\Phi_2$

$$= \frac{1}{R_1}\chi\left(t-\frac{R_1}{\beta}\right) + \frac{1}{R_2}\chi\left(t-\frac{R_2}{\beta}\right) \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

This is the solution in the case of a centre of rotation operating in a semi-infinite medium.

From (24), we find that the displacement at a point $(x, y, 0)$ on the surface $z = 0$ is

$$(u_\theta)_{z=0} = -\frac{2r}{R_0^3}\chi\left(t-\frac{R_0}{\beta}\right) - \frac{2r}{\beta R_0^2}\chi'\left(t-\frac{R_0}{\beta}\right) \quad \ldots \quad \ldots \quad (27)$$

where

$$R_0^2 = x^2+y^2+\frac{c^2}{k^2}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

4. *Some particular cases:* We consider below some particular cases :—

(*a*) Suppose $\chi(t) = \sin pt$

Then the displacement on the surface $z = 0$ is

$$(u_\theta)_{z=0} = -\frac{2r}{R_0^3}\sin p\left(t-\frac{R_0}{\beta}\right)-\frac{2rp}{\beta R_0^2}\cos p\left(t-\frac{R_0}{\beta}\right) \quad \ldots \quad \ldots \quad (29)$$

(*b*) If $\chi(t) = e^{-qt}\cos pt$ we find that the displacement on the surface $z = 0$ is

$$u_\theta = -\frac{2r}{R_0^3}\exp\left\{-q\left(t-\frac{R_0}{\beta}\right)\right\}\sin p\left(t-\frac{R_0}{\beta}\right)+$$

$$+\frac{2r}{\beta R_0^2}\exp\left\{-q\left(t-\frac{R_0}{\beta}\right)\right\}\left\{q\cos p\left(t-\frac{R_0}{\beta}\right)+p\sin p\left(t-\frac{R_0}{\beta}\right)\right\} \quad \ldots \quad (30)$$

(*c*) $\chi(t) = H(t)$ where $H(t)$ is Heaviside unit function defined by

$$H(t) = 0 \text{ for } t < 0$$

$$= 1 \text{ for } t > 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

From (27), we have the surface displacement in this case

$$(u_\theta)_{z=0} = -\frac{2r}{R_0^3}H\left(t-\frac{R_0}{\beta}\right)-\frac{2r}{\beta R_0^2}\delta\left(t-\frac{R_0}{\beta}\right) \quad \ldots \quad \ldots \quad (32)$$

where $\delta(t)$ is the Dirac delta function.

5. *Numerical calculations:* The values of the elastic constants of a transversely isotropic hexagonal crystal, Beryl, are (Love, 1927, Art. 113)

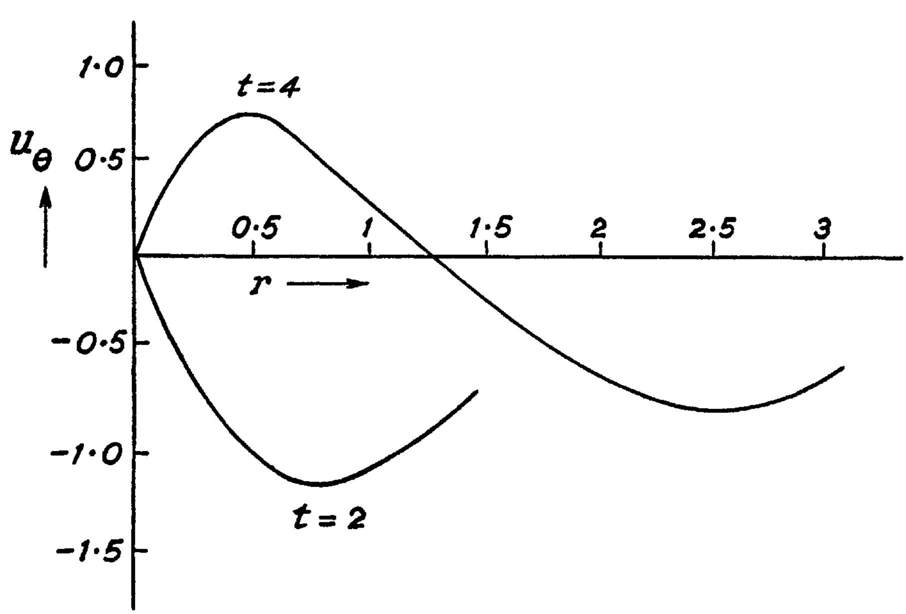

FIG. 1.

$$c_{11} = 2{\cdot}746 \times 10^9 \quad \text{gms. wt. per sq. cm.}$$
$$c_{33} = 2{\cdot}409 \times 10^9 \quad ,,$$
$$c_{12} = 0{\cdot}980 \times 10^9 \quad ,,$$
$$c_{13} = 0{\cdot}674 \times 10^9 \quad ,,$$
$$c_{44} = 0{\cdot}666 \times 10^9 \quad ,,$$
$$\text{Also } \rho = 2{\cdot}7 \text{ gms per cu. cm.}$$

Therefore the velocity of the waves in Beryl is

$$\beta = 8{\cdot}01 \text{ km./sec.}$$

Figure 1 shows surface displacement at different distances for times $t = 2$, $t = 4$ for the case of $\chi(t) = \sin t$ taking $\beta = 1$.

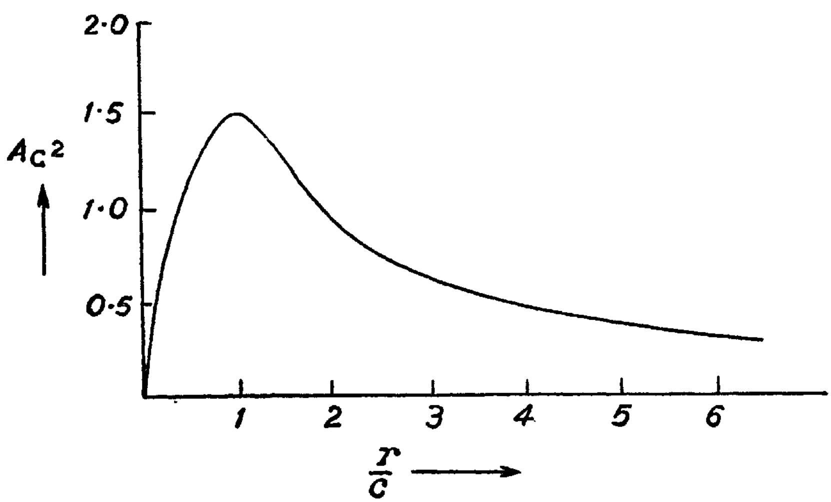

FIG. 2.

Figure 2 shows amplitude A of surface displacement for different distances (r/c) in the case of $\chi(t) = \sin pt$.

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