ON THE CESÀRO SUMMABILITY OF THE ULTRASPHERICAL SERIES (1)

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(Communicated by B. N. Prasad, F.N.I.)

(*Received June 10 ; read June 26, 1958*)

Abstract

The Cesàro summability of the ultraspherical series was studied in detail by Kogbetliantz (1923*a*, *b*, 1924). Later on some of his results were generalised by Obrechkoff (1936). In this paper, some new results on the (C, δ) summability of the ultraspherical series have been obtained in the case $\delta > \lambda$, λ being the order of the ultraspherical polynomials involved.

1. The ultraspherical polynomials $P_n^{(\lambda)}(x)$ are defined by the following expansion :—

$$(1.1) \qquad (1-2xt+t^2)^{-\lambda} = \sum_{n=0}^{\infty} t^n P_n^{(\lambda)}(x), \quad \lambda > 0.$$

If $f(\theta, \phi)$ be a function defined for the range $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$, the ultraspherical series corresponding to it on the sphere S is

$$(1.2) \qquad f(\theta, \phi) \sim \frac{1}{2\pi} \sum_{n=0}^{\infty} (n+\lambda) \int_S \int \frac{f(\theta', \phi') P_n^{(\lambda)}(\cos \omega) \sin \theta' \, d\theta' \, d\phi'}{[\sin^2\theta' \sin^2(\phi-\phi')]^{\frac{1}{2}-\lambda}},$$

where

$$\cos \omega = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi').$$

The Laplace series is a particular case of this series for $\lambda = \frac{1}{2}$, while this reduces to the trigonometric series in the limit as $\lambda \to 0$, because

$$(1.3) \qquad \lim_{\lambda \to 0} \frac{1}{\lambda} P_n^{(\lambda)}(\cos \theta) = \frac{2}{n} \cos n\theta, \quad n \geqslant 1.$$

A generalised mean value of $f(\theta, \phi)$ on the sphere has been defined by Kogbetliantz (1924) as follows :—

$$(1.4) \qquad f(\omega) = \frac{1}{2\pi (\sin \omega)^{2\lambda}} \int_{C_\omega} \frac{f(\theta', \phi') \, ds'}{[\sin^2\theta' \sin^2(\phi-\phi')]^{\frac{1}{2}-\lambda}},$$

where the integral is taken along the small circle whose centre is (θ, ϕ) on the sphere and whose curvilinear radius is ω.

It is assumed throughout that the function

$$(1.5) \qquad f(\theta', \phi') [\sin^2\theta' \sin^2(\phi-\phi')]^{\lambda-\frac{1}{2}}$$

is absolutely integrable (L) over the sphere S. In the case $k < 2\lambda$, k being the order of Cesàro sum, we also assume the integrability on the whole sphere of the function

$$(1.6) \qquad \left(\cos \frac{\omega}{2}\right)^{k-2\lambda} f(\theta', \phi') [\sin^2\theta' \sin^2(\phi-\phi')]^{\lambda-\frac{1}{2}}.$$

We write

$$(1.7) \quad \phi(\omega) = \left[f(\omega) - \frac{A\Gamma(\lambda)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\lambda)}\right](\sin \omega)^{2\lambda};$$

$$\Phi_p(x) = \frac{1}{\Gamma(p)}\int_0^x (x-t)^{p-1}\phi(t)\,dt;$$

$$\Phi_0(x) = \phi(x);$$

$$\phi_p(x) = \Gamma(p+1)\,x^{-p}\,\Phi_p(x), \quad p \geqslant 0;$$

and

$$\Phi_p(x) = \frac{d}{dx}\Phi_{p+1}(x), \quad -1 < p < 0.$$

Kogbetliantz (1923*a*, 1924) had made a detailed study of the summability problem for the ultraspherical series (1.2) and had obtained theorems for the Cesàro summability of the series under the condition of continuity of $f(\theta, \phi)$ over the sphere (1924). Obrechkoff (1936) generalised his results in the form of theorems which correspond to those of Verblunsky (1932) and Bosanquet (1930) in the case of Fourier series. The object of this paper is to obtain, for the Cesàro summability of the ultraspherical series (1.2), some new results in a different line, which correspond to those obtained by Wang (1943, 1947) for Fourier series.

We prove the following theorems :—

Theorem 1. *If*

$$\int_0^t \phi(u)\,du = o\left(t^{\frac{1+2\lambda}{\alpha}}\right) \textit{ for } 0 < \alpha < 1 \quad \textit{and} \quad 0 < \lambda < 1,$$

then the series (1.2) *is summable* $(C, \alpha+\lambda)$ *at the point* (θ, ϕ) *to the sum* A.

Theorem 2. *If*

$$\phi_\alpha(t) = o\left(\frac{t^{2\lambda}}{\log\frac{1}{t}}\right) \quad \textit{as } t \to +0 \textit{ for } \alpha > 0 \textit{ and } 0 < \lambda < 1,$$

then the series (1.2) *is summable* $(C, \alpha+\lambda)$ *at the point* (θ, ϕ) *to the sum* A.

2. In order to simplify the proofs of the theorems, it will be convenient first to estimate the orders of certain special functions.

We write $s_n^k(\omega)$ for the nth Cesàro mean of order k of the series

$$(2.1) \quad \sum (n+\lambda)\,P_n^{(\lambda)}(\cos\omega).$$

Then we have, for $\lambda > 0$ and $p \geqslant 0$,

$$(2.2) \quad \frac{d^p\{s_n^k(\omega)\}}{d\omega^p} = \begin{cases} O(n^{2\lambda+p+1}) & \text{for } 0 \leqslant \omega \leqslant \pi,\ k > 0; \\ O\left(\dfrac{n^{\lambda+p-k}}{\omega^{k+\lambda+1}}\right) + O\left(\dfrac{1}{n\omega^{2\lambda+2+p}}\right) & \text{for } 0 < \omega \leqslant a < \pi; \\ O\left(\dfrac{n^{\lambda+p-k}}{\omega^{k+\lambda+1}}\right) & \text{for } 0 < \omega \leqslant a < \pi \text{ and } \lambda+1+p > k. \end{cases}$$

These orders have been evaluated by Obrechkoff (1936) and may also be obtained by arguments given by Kogbetliantz (1924). Henceforward we shall denote

$$\frac{d^p\{s_n^k(\omega)\}}{d\omega^p} \text{ by } s_n^{(p)}(\omega).$$

Kogbetliantz (1924) has shown that the integral

$$\int_\delta^\pi f(\omega)\,\sin^{2\lambda}(\omega)\,s_n^k(\omega)\,d\omega$$

tends to zero under the conditions (1.5) and (1.6), for each δ such that $0 < \delta < \pi$ and for each $k > \lambda$. Using this result of localisation Obrechkoff (1936) has proved that for (C, k) summability of the series (1.2) to the sum A, it is sufficient that the integral

$$i = \int_0^\delta \phi(\omega)\,s_n^k(\omega)\,d\omega \qquad (2.3)$$

should tend to zero for $0 < \delta < \pi$ and for each $k > \lambda$.

3. *Proof of theorem 1*:—In order to prove the theorem, it is sufficient to show that

$$\lim_{n\to\infty} \int_0^\delta \phi(u)\,s_n^{\alpha+\lambda}(u)\,du = 0 \qquad (3.1)$$

under the conditions of the theorem.

Let $\Delta = \dfrac{\alpha}{\alpha+2\lambda+1}$, and suppose $\eta = \dfrac{\mu}{n^\Delta}$, where μ is a constant taken sufficiently large.

We have

$$\begin{aligned}
&\int_0^\eta \phi(u)\,s_n^{\alpha+\lambda}(u)\,du \\
&= \left[s_n^{\alpha+\lambda}(u)\,\Phi_1(u)\right]_0^\eta - \int_0^\eta \Phi_1(u)\,s_n^{(1)}(u)\,du \\
&= J_1 - J_2, \text{ say.}
\end{aligned}$$

Now

$$\begin{aligned}
|J_1| &= o\left[\eta^{\frac{2\lambda+1}{\alpha}}\,n^{-\alpha}\,\eta^{-\alpha-2\lambda-1}\right] \\
&= o\left[n^{-\alpha+\Delta(\alpha+2\lambda+1)}\,\mu^{(2\lambda+1)\left(\frac{1}{\alpha}-1\right)-\alpha}\,n^{-\Delta\left(\frac{1+2\lambda}{\alpha}\right)}\right] \\
&= o\left[\mu^{(2\lambda+1)\left(\frac{1}{\alpha}-1\right)-\alpha}\,n^{-\Delta\left(\frac{1+2\lambda}{\alpha}\right)}\right] \\
&= o(1) \quad \text{as } n\to\infty.
\end{aligned}$$

Also

$$\begin{aligned}
J_2 &= \int_0^\eta \Phi_1(u)\,s_n^{(1)}(u)\,du \\
&= \int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^\eta = L_1 + L_2, \text{ say.}
\end{aligned}$$

Here, using (2.2)

$$|L_1| = O\left[\int_0^{\frac{1}{n}} |\Phi_1(u)|\, n^{2\lambda+2}\, du\right]$$

$$= o\left[\int_0^{\frac{1}{n}} u^{\frac{2\lambda+1}{\alpha}} n^{2\lambda+2}\, du\right]$$

$$= o\left[n^{(2\lambda+1)\left(1-\frac{1}{\alpha}\right)}\right]$$

$$= o(1), \quad \text{as } n \to \infty.$$

Again

$$|L_2| = \int_{\frac{1}{n}}^{\eta} O\left[\frac{n^{1-\alpha}}{u^{\alpha+2\lambda+1}}\right] . \, o\left(u^{\frac{2\lambda+1}{\alpha}}\right) du$$

$$= o\left[n^{1-\alpha}\int_{\frac{1}{n}}^{\eta} u^{(2\lambda+1)\left(\frac{1}{\alpha}-1\right)-\alpha}\, du\right]$$

$$= o\left[n^{1-\alpha}\, \eta^{\frac{2\lambda+1}{\alpha}-2\lambda-\alpha}\right] + o(1)$$

$$= o\left[n^{1-\alpha-\Delta\left(\frac{2\lambda+1}{\alpha}-\alpha-2\lambda\right)}\, \mu^{\frac{2\lambda+1}{\alpha}-2\lambda-\alpha}\right] + o(1)$$

$$= o\left[\mu^{\frac{2\lambda+1}{\alpha}-2\lambda-\alpha}\right] + o(1)$$

$$= o(1), \quad \text{as } n \to \infty.$$

Thus to establish (3.1) completely, it will be sufficient to show that

$$\operatorname*{Lim}_{\mu\to\infty} \overline{\operatorname*{Lim}_{n\to\infty}} \left|\int_{\eta}^{\delta} \phi(u)\, s_n^{\alpha+\lambda}(u)\, du\right| = 0,$$

where $\eta = \dfrac{\mu}{n^{\Delta}}$.

Now

$$\left|\int_{\eta}^{\delta} \phi(u)\, s_n^{\alpha+\lambda}(u)\, du\right| = O\left[\int_{\eta}^{\delta} |\phi(u)|\, \frac{n^{-\alpha}}{u^{\alpha+2\lambda+1}}\, du\right]$$

$$= O\left[n^{-\alpha} . \, n^{\Delta(\alpha+2\lambda+1)}\, \mu^{-(\alpha+2\lambda+1)} \int_0^{\pi} |\phi(u)|\, du\right]$$

$$= O\left[\mu^{-(\alpha+2\lambda+1)}\right]$$

$$= o(1), \quad \text{as } \mu \to \infty.$$

This completes the proof of the theorem.

4. *Proof of theorem 2* :—Let m be the integer such that $\alpha \leqslant m < \alpha+1$. Then

$$i = \int_0^{\delta} \phi(\omega)\, s_n^k(\omega)\, d\omega$$

$$= \left[\sum_{\rho=1}^{m} (-1)^{\rho-1} \Phi_\rho(\omega) \left(\frac{d}{d\omega}\right)^{\rho-1} s_n^{\alpha+\lambda}(\omega) \right]_0^{\delta} + (-1)^m \int_0^{\delta} \Phi_m(t) \left(\frac{d}{dt}\right)^m s_n^{\alpha+\lambda}(t)\, dt.$$

From (2.2) we observe that

$$s_n^{(q)}(\delta) = O(n^{q+\lambda-\alpha-\lambda}) + O(n^{-1})$$
$$= o(1), \quad \text{as } n \to \infty,$$

provided that $\alpha > q$.

Thus

$$i = o(1) + (-1)^m \int_0^{\delta} \Phi_m(t)\, s_n^{(m)}(t)\, dt$$

(4.1) $$= o(1) + (-1)^m J.$$

We write

$$J = \int_0^{\delta} \Phi_m(t)\, s_n^{(m)}(t)\, dt$$

$$= \int_0^{\frac{1}{n^r}} + \int_{\frac{1}{n^r}}^{\delta} = J_1 + J_2,$$

where $r = \dfrac{\alpha - m + 1}{\alpha + 2\lambda + 1}$.

Then, since $\alpha > 0$, we observe that $m \geqslant 1$ and thus $\Phi_m(t)$ is absolutely continuous. We now have

$$J_2 = \int_{\frac{1}{n^r}}^{\delta} \Phi_m(t)\, s_n^{(m)}(t)\, dt$$

$$= \left[\Phi_m(t)\, s_n^{(m-1)}(t) \right]_{\frac{1}{n^r}}^{\delta} - \int_{\frac{1}{n^r}}^{\delta} \Phi_{m-1}(t)\, s_n^{(m-1)}(t)\, dt$$

$$= R - S, \text{ say.}$$

As $\Phi_m(t) = o(1)$, we have

$$R = \Phi_m(\delta)\, s_n^{(m-1)}(\delta) + o(1)\,.\, s_n^{(m-1)}\left(\frac{1}{n^r}\right)$$

$$= o(1)\,.\, n^{m-\alpha-1} + o(1)\,.\, O[n^{r(\alpha+2\lambda+1)+m-1-\alpha}], \text{ by (2.2)},$$

(4.2) $$= o(1), \quad \text{as } n \to \infty.$$

Also, if we write

$$\Phi^*(t) = \int_0^t |\Phi_{m-1}(u)|\, du,$$

then

$$S = O\left[n^{m-1-\alpha} \int_{\frac{1}{n^r}}^{\delta} \frac{|\Phi_{m-1}(t)|\, dt}{t^{\alpha+2\lambda+1}}\right]$$

$$= O(n^{m-1-\alpha}) \left[\frac{\Phi^*(t)}{t^{\alpha+2\lambda+1}}\right]_{\frac{1}{n^r}}^{\delta} + O(n^{m-1-\alpha}) \int_{\frac{1}{n^r}}^{\delta} \frac{\Phi^*(t)}{t^{\alpha+2\lambda+2}}\, dt$$

$$= O(n^{m-1-\alpha}) + O(n^{m-1-\alpha}) \,.\, o\{n^{r(\alpha+2\lambda+1)}\} + O(n^{m-1-\alpha}) \int_{\frac{1}{n^r}}^{\delta} o(t^{-\alpha-2\lambda-2})\, dt$$

$$= o(1), \quad \text{as } n \to \infty,$$

since $\Phi^*(t) = o(1)$.

When α is not an integer,

$$J_1 = \int_0^{\frac{1}{n^r}} \Phi_m(t)\, s_n^{(m)}(t)\, dt$$

$$= \int_0^{\frac{1}{n^r}} s_n^{(m)}(t)\, dt\, \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-u)^{m-\alpha-1}\, \Phi_\alpha(u)\, du$$

$$(4.3) \qquad = \frac{1}{\Gamma(m-\alpha)} \int_0^{\frac{1}{n^r}} \Phi_\alpha(u)\, du \int_u^{\frac{1}{n^r}} (t-u)^{m-\alpha-1}\, s_n^{(m)}(t)\, dt,$$

by Fubini's theorem, since $\Phi_\alpha(u) = O(u^\alpha)$ and $m > \alpha$.

Hence

$$J_1 = \int_0^{\frac{1}{n^r}} \Phi_\alpha(u)\, F(n^{-r}, u)\, du,$$

where
$$F(n^{-r}, u) = \frac{1}{\Gamma(m-\alpha)} \int_u^{n^{-r}} (t-u)^{m-\alpha-1}\, s_n^{(m)}(t)\, dt.$$

We now evaluate the order of $F(n^{-r}, u)$.
If $2u < n^{-r}$,

$$F(n^{-r}, u) = \left[\int_u^{2u} + \int_{2u}^{n^{-r}}\right] (t-u)^{m-\alpha-1} s_n^{(m)}(t)\, dt$$

$$= \int_u^{2u} (t-u)^{m-\alpha-1} O(n^{2\lambda+m+1})\, dt + u^{m-\alpha-1} \int_{2u}^{\xi} s_n^{(m)}(t)\, dt,$$

using (2.2) and the mean value theorem, ξ being given by $2u \leqslant \xi \leqslant n^{-r}$.
Thus

$$F(n^{-r}, u) = O(n^{2\lambda+m+1}\, u^{m-\alpha}) + u^{m-\alpha-1}\, O(n^{2\lambda+m})$$

$$= O(n^{2\lambda+m+1}\, u^{m-\alpha}) + O(u^{m-\alpha-1}\, n^{2\lambda+m}). \tag{4.4}$$

Obviously there will be no need to break the integral into two parts in case $2u > n^{-r}$.

Again, if $u+\frac{1}{n} < n^{-r}$

$$F(n^{-r}, u) = \int_u^{u+\frac{1}{n}} + \int_{u+\frac{1}{n}}^{n^{-r}} = F_1 + F_2\text{, say.}$$

$$F_1 = O\left\{\int_u^{u+\frac{1}{n}} n^{m-\alpha}\,.\, \frac{(t-u)^{m-\alpha-1}}{t^{\alpha+2\lambda+1}}\, dt\right\}$$

$$= O(n^{m-\alpha})\left\{u^{-\alpha-2\lambda-1} \int_u^{u+\frac{1}{n}} (t-u)^{m-\alpha-1}\, dt\right\}$$

$$= O(u^{-\alpha-2\lambda-1}),$$

and

$$F_2 = \int_{u+\frac{1}{n}}^{n^{-r}} (t-u)^{m-\alpha-1} s_n^{(m)}(t)\, dt$$

$$= n^{\alpha+1-m} \int_{u+\frac{1}{n}}^{\mu} s_n^{(m)}(t)\, dt, \quad u+\frac{1}{n} \leqslant \mu \leqslant n^{-r},$$

$$= n^{\alpha+1-m} \left[s_n^{(m-1)}(t)\right]_{u+\frac{1}{n}}^{\mu}$$

$$= O(\mu^{-\alpha-2\lambda-1}) + O(u^{-\alpha-2\lambda-1}) = O(u^{-\alpha-2\lambda-1}).$$

Thus

$$F(n^{-r}, u) = O(u^{-\alpha-2\lambda-1}). \tag{4.5}$$

Consequently, now

$$J_1 = \int_0^{\frac{1}{n^r}} \Phi_\alpha(u)\, F(n^{-r}, u)\, du$$

$$= \int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\frac{1}{n^r}}$$

$$= T + U, \text{ say.}$$

Now

$$T = \int_0^{\frac{1}{n}} \Phi_\alpha(u)\, F(n^{-r}, u)\, du$$

$$= o\left[\int_0^{\frac{1}{n}} \frac{u^{2\lambda+\alpha}}{\log \frac{1}{u}} u^{m-\alpha} \,.\, n^{2\lambda+m+1}\, du\right] + o\left[\int_0^{\frac{1}{n}} \frac{u^{2\lambda+\alpha}}{\log \frac{1}{u}} u^{m-\alpha-1} n^{2\lambda+m}\, du\right] \text{ by (4.4)}$$

$$= o(1).$$

Again

$$U = \int_{\frac{1}{n}}^{\frac{1}{n^r}} \Phi_\alpha(u)\, F(n^{-r}, u)\, du$$

$$= o\left[\int_{\frac{1}{n}}^{\frac{1}{n^r}} \frac{u^{2\lambda+\alpha}}{\log \frac{1}{u}} \frac{1}{u^{\alpha+2\lambda+1}}\, du\right] \text{ using (4.5)}$$

$$= o\left[\int_{\frac{1}{n}}^{\frac{1}{n^r}} \frac{du}{u \log \frac{1}{u}}\right]$$

$$= o\left(\log \frac{1}{r}\right) = o(1).$$

When $\alpha = m$, an integer,

$$J_1 = \int_0^{\frac{1}{n^r}} \Phi_\alpha(t)\, s_n^{(\alpha)}(t)\, dt$$

$$= \int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\frac{1}{n^r}} = V + W, \text{ say.}$$

Here

$$V = \int_0^{\frac{1}{n}} \Phi_\alpha(t)\, s_n^{(\alpha)}(t)\, dt$$

$$= o\left[\int_0^{\frac{1}{n}} \frac{u^{2\lambda+\alpha}}{\log \frac{1}{u}} \,.\, n^{2\lambda+\alpha+1}\, du\right]$$

$$= o(1).$$

Also

$$W = \int_{\frac{1}{n}}^{\frac{1}{n^r}} \Phi_\alpha(t)\, s_n^{(\alpha)}(t)\, dt$$

$$= \int_{\frac{1}{n}}^{\frac{1}{n^r}} o\left\{\frac{u^{2\lambda+\alpha}}{\log \frac{1}{u}}\right\} . \, O\left(\frac{1}{u^{\alpha+2\lambda+1}}\right) du$$

$$= o\left(\int_{\frac{1}{n}}^{\frac{1}{n^r}} \frac{du}{u \log \frac{1}{u}}\right)$$

$$= o\left(\log \frac{1}{r}\right) = o(1).$$

This completes the proof.

Acknowledgement

The author would like to acknowledge his indebtedness to Dr. B. N. Prasad for his valuable guidance during the preparation of this paper.

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