THE GENERAL THEORY OF MODERATED CHARGES—II

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Abstract

In the present paper, some cases where for a moderated charge an equivalent charge in the strict sense exists have been examined and when it does not exist, satisfactory average values have been obtained. In particular, it has been shown that for motion after all-burnt, an equivalent charge 'in the strict sense' always exists.

1. Introduction

The Equivalent Charge Method in the General Theory of Moderated Charges was developed in a recent paper by the present author (Kapur, 1957*a*). The present paper is a continuation of that paper and consequently that paper will be referred to as Part I.

We defined there the equivalent charge as that charge which would give the same ballistic equations as the moderated charge both during and after the burning of the charge. From this definition, we tried to deduce the shape, size and composition of the equivalent charge. However, since the shape, size and composition are defined by means of a number of parameters, viz. the force constant F, the charge mass C, the ballistic size D, the rate of burning constant β, the form-factor θ, the ratio of specific heats γ, the propellant density δ, the covolume per unit mass b, it is possible that all these may not be determined from the four basic equations of Internal Ballistics. Actually, we find that while all these can be determined, some of these, in general, vary throughout the burning period except under certain plausible assumptions which reduce these varying quantities to constants. After all-burnt, all the parameters are found to be constant, and we say that an equivalent charge always exists 'in the strict sense' for motion after all-burnt.

In the present paper, we investigate the conditions for the existence of an equivalent charge in the strict sense. When these conditions are not satisfied, two alternative courses are open to us:

(i) We may try to find satisfactory average values of the varying parameters,

or

(ii) We may integrate the equations of Internal Ballistics, taking such variations into account.

Both these methods have been investigated in the present paper.

2. The Fundamental Equations

When the rth component charge is burning, the basic equations for the moderated charge are:

$$\frac{F_1C_1}{\gamma_1-1}+\frac{F_2C_2}{\gamma_2-1}+\ldots\ldots+\frac{F_rC_r}{\gamma_r-1}z_r$$
$$=p\left\{K_0-\frac{C_r}{\delta_r}(1-z_r)-\frac{C_{r+1}}{\delta_{r+1}}-\ldots-\frac{C_n}{\delta_n}\right.$$
$$\left.-C_1b_1-C_2b_2-\ldots-C_{r-1}b_{r-1}-C_rb_rz_r\right\}$$
$$\times\left\{\frac{\dfrac{m_1C_1}{\gamma_1-1}+\ldots+\dfrac{m_{r-1}C_{r-1}}{\gamma_{r-1}-1}+\dfrac{m_rC_r}{\gamma_r-1}z_r}{m_1C_1+m_2C_2+\ldots+m_rC_rz_r}\right\}$$
$$+\tfrac{1}{2}[1{\cdot}05W+\tfrac{1}{3}(C_1+C_2+\ldots+C_{r-1}+C_rz_r)]v^2 \quad \ldots \quad \ldots \quad (1)$$

$$D_r\frac{df_r}{dt}=-\beta_rp \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$z_r=(1-f_r)(1+\theta_rf_r) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

$$[1{\cdot}05W+\tfrac{1}{3}(C_1+C_2+\ldots+C_rz_r)]\frac{d^2x}{dt^2}=Ap \quad \ldots \quad \ldots \quad (4)$$

The corresponding equations for the equivalent charge are:

$$\frac{FC}{\gamma-1}z=p\left[K_0-\frac{C}{\delta}(1-z)-Cbz\right]\frac{1}{\gamma-1}+\tfrac{1}{2}[1{\cdot}05w+\tfrac{1}{3}Cz]v^2 \quad \ldots \quad (5)$$

$$D\frac{df}{dt}=-Bp \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

$$z=\phi_r(f) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

$$[1{\cdot}05W+\tfrac{1}{3}Cz]\frac{d^2x}{dt^2}=Ap \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

The two equations (1) and (5) would become identical if

$$\frac{FCz}{\gamma-1}=\frac{F_1C_1}{\gamma_1-1}+\frac{F_2C_2}{\gamma_2-1}+\ldots+\frac{F_rC_r}{\gamma_r-1}z_r \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

$$Cz=C_1+C_2+\ldots+C_{r-1}+C_rz_r \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

$$\frac{C}{\delta}=\frac{C_1}{\delta_1}+\frac{C_2}{\delta_2}+\ldots+\frac{C_n}{\delta_n} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10a)$$

$$\frac{m_1C_1}{\gamma_1-1}+\frac{m_2C_2}{\gamma_2-1}+\ldots+\frac{m_{r-1}C_{r-1}}{\gamma_{r-1}-1}+\frac{m_rC_r}{\gamma_r-1}z_r=\frac{m_1C_1}{\gamma-1}+\ldots+\frac{m_rC_rz_r}{\gamma-1} \quad \ldots \quad (11)$$

$$C_1\left(b_1-\frac{1}{\delta_1}\right)+C_2\left(b_2-\frac{1}{\delta_2}\right)+\ldots+C_{r-1}\left(b_{r-1}-\frac{1}{\delta_{r-1}}\right)+C_r\left(b_r-\frac{1}{\delta_r}\right)z_r$$
$$=Cz\left(b-\frac{1}{\delta}\right) \quad \ldots \quad (12)$$

The two equations (4) and (8) will be satisfied if (10) is satisfied.

From (2) and (6)

$$\frac{1}{\beta_r'}\frac{df_r}{dt} = \frac{1}{\beta'}\frac{df}{dt}, \quad \ldots \quad (13)$$

where

$$\beta_r' = \frac{\beta_r}{D_r}, \quad \beta' = \frac{\beta}{D}. \quad \ldots \quad (14)$$

Integrating (13)

$$\frac{1-f_r}{\beta_r'} = \frac{f_{[r-1]}-f}{\beta'}, \quad \ldots \quad (15)$$

where $f_{[r-1]}$ is the value of f at the end of the $(r-1)$th stage, and from (15)

$$f_{[r]} = f_{[r-1]} - \frac{1}{k_r},$$

where

$$k_r = \frac{\beta_r'}{\beta'}, \quad \ldots \quad (16)$$

so that

$$f_{[r]} = 1 - \frac{1}{k_1} - \frac{1}{k_2} \ldots - \frac{1}{k_r}. \quad \ldots \quad (17)$$

Since $f_{[n]} = 0$, we get

$$\left.\begin{aligned} &\frac{1}{k_1}+\frac{1}{k_2}+\ldots+\frac{1}{k_n} = 1, \\ \text{or} \quad &\frac{1}{\beta_1'}+\frac{1}{\beta_2'}+\ldots+\frac{1}{\beta_n'} = \frac{1}{\beta'} \\ \text{or} \quad &\frac{D_1}{\beta_1}+\frac{D_2}{\beta_2}+\ldots+\frac{D_n}{\beta_n} = \frac{D}{\beta}. \end{aligned}\right\} \quad \ldots \quad (18)$$

From (3), (9) and (10)

$$z = A_r + B_r(1-f) - E_r(1-f)^2 \quad \ldots \quad (19)$$

where

$$A_r = \lambda_1+\lambda_2+\ldots+\lambda_{r-1}+\lambda_r(1-b_r)(1+\theta_r b_r) \quad \ldots \quad (20)$$

$$B_r = \lambda_r k_r[1-\theta_r+2\theta_r b_r] \quad \ldots \quad (21)$$

$$E_r = \lambda_r k_r^2 \theta_r \quad \ldots \quad (22)$$

$$\lambda_i = \frac{\dfrac{F_i C_i}{\gamma_i - 1}}{\dfrac{FC}{\gamma - 1}} \; [i=1, 2, \ldots n] \quad \ldots \quad (23)$$

$$b_r = \frac{\dfrac{1}{k_1}+\dfrac{1}{k_2}+\ldots+\dfrac{1}{k_r}}{\dfrac{1}{k_r}}, \; [r = 1, 2, \ldots n] \quad \ldots \quad (24)$$

(19) was the form-function deduced by us in Part I.
Now let us examine each of three equations (10), (11) and (12).

(*a*) Equation (10) expresses that the kinetic energy of the propellant gases is the same for the moderated and the equivalent charge. Even in the Internal Ballistics of a single charge, taking this kinetic energy as variable introduces certain complications and, therefore, the assumption is always made there that the unburnt propellant moves with the gases and in this way, both in (5) and (8), we use $\frac{1}{3}C$ instead of $\frac{1}{3}Cz$. If we make the same assumption here, equation (10) would simply require that

$$C = C_1+C_2+\ldots+C_n \quad \ldots \quad (25)$$

In this case, however, it is possible to modify the usual assumption by simply assuming that rth unburnt component charge moves with the gases so that we replace $\frac{1}{3}Cz$ both in (5) and (8) by $\frac{1}{3}(C_1+C_2+\cdots+C_r)$ during the rth stage of burning.

(*b*) Equation (11) determines the value of γ during the rth stage. Since, however, z_r varies from 0 to 1 during the rth stage, the value of γ will not be a constant. However, a satisfactory value for the rth stage is given by

$$\frac{1}{\gamma-1} = \frac{\sum_{i=1}^{r} \frac{m_i C_i}{\gamma_i - 1}}{\sum_{i=1}^{r} m_i C_i} \quad \ldots \quad (26)$$

(*c*) Equation (12) determines the value of $b-\frac{1}{\delta}$ during the rth stage. Again this is, in general, variable and we shall find a satisfactory average value for this in section 4.

3. Conditions or Assumptions under which an Equivalent Charge in the strict sense exists

Case I: Let us assume

$$\text{(i)}\ \gamma_1 = \gamma_2 = \cdots = \gamma_n$$

$$\text{(ii)}\ b_1-\frac{1}{\delta_1} = b_2-\frac{1}{\delta_2} = \cdots = b_n-\frac{1}{\delta_n} = 0 \quad \ldots \quad (27)$$

These are the usual assumptions made in the theory of composite or moderated charges. The first is justified by the fact that most of the propellants used in practice have practically the same value for the ratio of specific heats. The second is the usual assumption made even for a single charge since it considerably simplifies the solution of the equations of Internal Ballistics. Of course, it is justified by the fact that for the propellants used in practice, the difference between specific volume and co-volume is very small.

Under the assumptions (27), all the parameters for the equivalent charge come out to be constant. γ is the common value in (27*a*), $b = \frac{1}{\delta}$ from (27*b*), δ is determined from (10*a*), C is determined from (25), F is determined from

$$FC = F_1C_1+F_2C_2+\ldots+F_nC_n \quad \ldots \quad (28)$$

and the form-function is determined as in (19).

Case II :

Here we assume

$$\text{(i)}\quad \gamma_1 = \gamma_2 = \ldots = \gamma_n = \gamma \qquad (29a)$$

$$\text{(ii)}\quad \frac{F_1}{b_1 - \frac{1}{\delta_1}} = \frac{F_2}{b_2 - \frac{1}{\delta_2}} = \ldots = \frac{F_n}{b_n - \frac{1}{\delta_n}} = \frac{F}{b - \frac{1}{\delta}} \qquad (29b)$$

In this case (9) and (12) become identical and we determine C from (25), γ from (28*a*), F from (28), δ from (10*a*), b from (28*b*), and form-function from (10).

Case III : Motion after all-burnt :

For motion after all-burnt

$$\frac{FC}{\gamma - 1} = \frac{F_1 C_1}{\gamma_1 - 1} + \ldots + \frac{F_n C_n}{\gamma_n - 1} \qquad (30)$$

$$\sum_{i=1}^{n} \frac{m_i C_i}{\gamma_i - 1} = \frac{1}{\gamma - 1} \sum_{i=1}^{n} m_i C_i \qquad (31)$$

$$C\left(b - \frac{1}{\delta}\right) = \sum_{i=1}^{n} C_i \left(b_i - \frac{1}{\delta_i}\right), \qquad (32)$$

so that all the parameters are constant, and an equivalent charge in the strict sense exists.

4. Average Values of $b - \frac{1}{\delta}$ and γ

Using Corner's method for fitting in the case of composite charges as adapted by Kapur (1956), we choose $b - \frac{1}{\delta}$ so that

$$\sum_{r=1}^{n} \int_{1-\frac{b_{r-1}}{k_{r-1}}}^{1-\frac{b_r}{k_r}} [A_r' + B_r'(1-f) - E_r'(1-f)^2]\, df$$

$$= \sum_{r=1}^{n} \int_{1-\frac{b_{r-1}}{k_{r-1}}}^{1-\frac{b_r}{k_r}} [A_r + B_r(1-f) - E_r(1-f)^2]\, df \qquad (33)$$

where A_r', B_r', E_r' are obtained from A_r, B_r, E_r by replacing λ_i by λ_i', where

$$\lambda_i' = \frac{C_i\left(b_i - \frac{1}{\delta_i}\right)}{C\left(b - \frac{1}{\delta}\right)} \quad [i = 1, 2, \ldots, n] \qquad (34)$$

$$\therefore \sum_{r=1}^{n} A_r' \left(\frac{b_{r-1}}{k_{r-1}} - \frac{b_r}{k_r}\right) + \sum_{r=1}^{n} \frac{B_r'}{2} \left(\frac{b_{r-1}^2}{k_{r-1}^2} - \frac{b_r^2}{k_r^2}\right)$$

$$- \sum_{r=1}^{n} \frac{E_r'}{3} \left(\frac{b_{r-1}^3}{k_{r-1}^3} - \frac{b_r^3}{k_r^3}\right) = \sum_{r=1}^{n} A_r \left(\frac{b_{r-1}}{k_{r-1}} - \frac{b_r}{k_r}\right)$$

$$+ \sum_{r=1}^{n} \frac{B_r}{2} \left(\frac{b_{r-1}^2}{k_{r-1}^2} - \frac{b_r^2}{k_r^2}\right) - \sum_{r=1}^{n} \frac{E_r}{3} \left(\frac{b_{r-1}^3}{k_{r-1}^3} - \frac{b_r^3}{k_r^3}\right) \quad \ldots \quad \ldots \quad (35)$$

In this case, it is easily verified that the present method gives the same result as the Method of Least Squares.

For estimating the average value of γ, we have

$$\sum_{r=1}^{n} \int_{1-\frac{b_{r-1}}{k_{r-1}}}^{1-\frac{b_r}{k_r}} [A_r'' + B_r''(1-f) - E_r''(1-f)^2] df$$

$$= \sum_{r=1}^{n} \int_{1-\frac{b_{r-1}}{k_{r-1}}}^{1-\frac{b_r}{k_r}} [A_r''' + B_r'''(1-f) - E_r'''(1-f)^2] df, \quad \ldots \quad (36)$$

where A_r'', B_r'', E_r'' ; A_r''', B_r''', E_r''' are obtained from A_r, B_r, E_r by replacing λ_i by

$$\lambda_i'' = \frac{\dfrac{m_i C_i}{\gamma_i - 1}}{\dfrac{mC}{\gamma - 1}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (37a)$$

$$\lambda_i''' = \frac{m_i C_i}{mC} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (37b)$$

This average has to be used if we want the same γ throughout burning, but in practice, it is not necessary to insist on the same γ; since the integration has to be carried out stage by stage. During the rth stage, we can use the average given by (26). This will also give the correct value for motion after all-burnt.

5. Integration of the Equations in the most General Case

We can integrate the equations of Internal Ballistics taking the variation of b and γ into account. The integration is carried out in exactly the same way as for a composite charge and the discussion given by Kapur (1957*b*) for composite charges applies here completely except that the values of A_r, B_r, E_r; A_r', B_r', E_r'; A_r'', B_r'', E_r'' ; A_r''', B_r''', E_r''' and the corresponding values of k_r, k_r', k_r'', k_r''' ; a_r, a_r', a_r'', a_r''' and b_r, b_r', b_r'', b_r''' have to be modified to the values given in the present paper.

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