EFFECTS OF VARIATIONS OF LOADING CONDITIONS ON INTERNAL BALLISTICS—II

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ABSTRACT

In an earlier paper [Kapur (1959*a*), Part I of the present paper], it had been shown that the 36 partial derivatives of p_1, x_2 and v_3 of first rank could be expressed in terms of 9 basic partial derivatives. In the present paper, it has been shown that all the 792 partial derivatives of the second rank and all the 7,920 partial derivatives of the third rank of these variables can also be expressed in terms of these 9 basic partial derivatives. It has been shown that pv has a unique maximum for composite charges and the conditions for it to occur in or at the end of any stage have been determined. It has also been shown that when the shot-start pressure is not zero, the velocity-time and pressure-time relations can be expressed in terms of incomplete Beta functions.

1. INTRODUCTION

Let $X_{i,y}$ denote the partial derivative $\dfrac{\partial(\log X_i)}{\partial(\log y)}$, where X_i is the value of the variable X at the position corresponding to suffix i, X may be any one of the variables: x (shot-travel), v (velocity), p (pressure); i may be any one of the three suffixes: 1 (corresponding to maximum pressure), 2 (corresponding to all-burnt position), 3 (corresponding to muzzle); and y may stand for any one of the twelve loading conditions : F (force-constant), C (charge mass), D (ballistic size), β (rate of burning constant), θ (form-factor), b (co-volume per unit mass), δ (propellant density), K_0 (chamber capacity), A (area of cross-section of gun barrel), x_3 (shot-travel up to muzzle), w (shot-mass), p_0 (shot-start pressure). In a recent paper [Kapur (1959*a*), hereinafter referred to as I], ballistic similitudes have been used to express all the partial derivatives of maximum pressure p_1 in terms of $p_{1,D}$; p_{1,p_0} ; $p_{1,b}$. Similarly all the derivatives of the shot-travel till all-burnt x_2 have been expressed in terms of $x_{2,D}$; x_{2,p_0} ; $x_{2,b}$ and all the partial derivatives of the muzzle velocity v_3 have been expressed in terms of $v_{3,D}$; v_{3,p_0} ; $v_{3,b}$. If b and p_0 are kept fixed, the partial derivatives can be expressed in terms of $p_{1,D}$; $x_{2,D}$; $v_{3,D}$ alone. All these three have been tabulated in I. The first and third of these have also been tabulated by Tawakley (1959), [hereinafter referred to as II].

In II, explicit formulae have been given for $p_{1,\beta}$; $p_{1,D}$; $p_{1,C}$; $p_{1,F}$; $p_{1,K0}$; $p_{1,W}$; $v_{3,\beta}$; $v_{3,D}$; $v_{3,C}$; $v_{3,F}$; $v_{3,W}$; but it is obvious that explicit formulae for not only these 11 but for the other 25 derivatives also can be written in terms of the 9 basic partial derivatives: $p_{1,D}$; p_{1,p_0} ; $p_{1,b}$; $x_{2,D}$; x_{2,p_0}; $x_{2,b}$; $v_{3,D}$; v_{3,p_0} ; $v_{3,b}$ in the general case and in terms of only 3 partial derivatives $p_{1,D}$; $x_{2,D}$; $v_{3,D}$ in the restricted case considered in II, with the help of formulae deduced in I.

Let $X_{i,y}$; X'_j, y' denote the partial derivative. $\frac{\partial(\log X_i)}{\partial(\log y)}$ when X'_j is kept constant by adjusting the loading condition y'. In II explicit formulae for the following 11 derivatives are given:

$$\begin{array}{lllllll} v_{3,C}\,;\,p_{1,D} & : & v_{3,D}\,;\,p_{1,C} & : & v_{3,K_0}\,;\,p_{1,D} & : & v_{3,F}\,;\,p_{1,b} \\ v_{3,W}\,;\,p_{1,C} & : & v_{3,K_0}\,;\,p_{1,C} & : & p_{1,C}\,;\,v_{3,D} & : & p_{1,F}\,;\,v_{3,D} \\ p_{1,K_0}\,;\,v_{3,D} & : & p_{1,W}\,;\,v_{3,D} & : & p_{1,\,x_3}\,;\,v_{3,D} & & \end{array}$$

It is obvious, however, from the lengthy expressions for these, even in the special case of zero shot-start pressure and neglect of co-volume correction terms, that direct tabulation of these would be extremely tedious. Moreover, the total number of such partial derivatives would be $3\times 12\times 2\times 11 = 792$, and although all these may not be of equal interest, the writing of explicit formulae for all these (in the general case each formula would take one full page) and preparation for their tables from these would be a Herculean task. Fortunately this is not necessary and all these 792 derivatives can be expressed in terms of the 9 basic partial derivatives, and once the tables for these 9 partial derivatives are prepared under the most general conditions, the tables for others can be prepared under the same general conditions without much difficulty, by using the simple formula deduced in section 2 and, of course, the formulae deduced in I.

The above partial derivatives may be called partial derivatives of the second rank. We may also define partial derivatives of the third rank $X_{i,y}$; X'_j, X''_K, y', y'' giving the value $\frac{\partial(\log X_i)}{\partial(\log y)}$ when X'_j and X''_K are kept constant by adjusting y' and y''. The total number of such derivatives would be $3\times 2\times 1\times 12\times 11\times 10 = 7920$, but it is shown in section 3 that all these can be expressed in terms of the 9 basic partial derivatives of the first rank.

In the case of second and third rank partial derivatives, it is of great interest to find variations in y' (or both y' and y'') necessary to keep X'_j (or both X'_j and X''_K) constant, corresponding to a given small change in y. It is shown in section 4 that these variations can also be expressed in terms of the 9 basic partial derivatives of the first rank. In II, the variation of pv which corresponds to the rate of change of K.E. has also been studied. In section 5, it has been shown that, in the case of composite charges, the maximum of pv is unique. A corresponding theorem for maximum of p alone had been established earlier by Kapur (1956).

In section 6, we have obtained the equation of the pressure-time curve. In II, it had been obtained for zero-shot-start pressure and tubular charge only. We show that in the more general case of non-zero-shot-start pressure and any value of form-factor θ, its equation can be expressed in terms of incomplete Beta functions tabulated by Pearson (1934). It is of interest to note that the importance of these functions in internal ballistics in another context has recently been pointed out by Kapur (1959*b*).

We may point out here that the rate of variation of pv had been studied earlier (Kapur, 1957) in view of the importance of this rate in the study of the effect of co-volume-correction terms on the uniqueness of maximum pressure (Kapur, 1956). He had proved that for a single charge pv goes on increasing till all-burnt if the central ballistic parameter

$$M < \frac{1}{f_0}\left[1+\frac{1}{f_0}-\theta\right]\frac{2}{3\gamma-1},$$

where f_0 is the value of f at shot-start.

2. PARTIAL DERIVATIVES OF SECOND RANK

Let

$$X_i = f(y, y', z) \quad \therefore \tag{1}$$

$$X'_j = \phi(y, y', z), \tag{2}$$

where z denotes the set of loading conditions which are not changed. Then since X'_j is kept constant,

$$\delta X_i = \frac{\partial X_i}{\partial y}\delta y + \frac{\partial X_i}{\partial y'}\delta y' \tag{3}$$

$$0 = \frac{\partial X'_j}{\partial y}\delta y + \frac{\partial X'_j}{\partial y'}\delta y' \tag{4}$$

$$\therefore \quad \delta X_i = \left[\frac{\partial X_i}{\partial y} - \frac{\partial X_i}{\partial y'}\frac{\dfrac{\partial X'_j}{\partial y}}{\dfrac{\partial X'_j}{\partial y'}}\right]\delta y \tag{5}$$

$$\therefore \quad \left[\frac{\partial(\log X_i)}{\partial(\log y)}\right]_{X'_j,\, y'} = \frac{\partial(\log X_i)}{\partial(\log y)} - \frac{\partial(\log X_i)}{\partial(\log y')}\frac{\dfrac{\partial(\log X'_j)}{\partial(\log y)}}{\dfrac{\partial(\log X'_j)}{\partial(\log y')}} \tag{6}$$

or

$$X_{i,\,y}\,;\,X'_{j,\,y'} = X_{i,\,y} - X_{i,\,y'} \times \frac{X'_{j,\,y}}{X'_{j,\,y'}} \tag{7}$$

This expresses any partial derivative of second rank in terms of partial derivatives of first rank, all of which can be expressed in terms of the 9 basic partial derivatives of first rank by formulae deduced in I.

3. PARTIAL DERIVATIVES OF THIRD RANK

Let

$$X_i = f(y, y', y'', z) \tag{8}$$

$$X'_j = \phi(y, y', y'', z) \tag{9}$$

$$X''_K = \psi(y, y', y'', z), \tag{10}$$

then since X'_j, X''_k, z remain constant

$$\delta X_i = \frac{\partial X_i}{\partial y}\delta y + \frac{\partial X_i}{\partial y'}\delta y' + \frac{\partial X_i}{\partial y''}\delta y'' \tag{11}$$

$$o = \frac{\partial X'_j}{\partial y}\delta y + \frac{\partial X'_j}{\partial y'}\delta y' + \frac{\partial X'_j}{\partial y''}\delta y'' \tag{12}$$

$$o = \frac{\partial X''_K}{\partial y}\delta y + \frac{\partial X''_K}{\partial y'}\delta y' + \frac{\partial X''_K}{\partial y''}\delta y'' \tag{13}$$

Eliminating $\delta y'$, $\delta y''$, we get

$$\begin{vmatrix} X_{i,y};\ X'_j, X''_{K,y',y''} - X_{i,y} & X_{i,y'} & X_{i,y''} \\ -X'_{j,y} & X'_{j,y'} & X'_{j,y''} \\ -X''_{K,y} & X''_{K,y'} & X''_{K,y''} \end{vmatrix} = 0$$

or

$$X_{i,y};\ X'_j, X''_{K,y',y''} = \frac{\begin{vmatrix} X_{i,y} & X_{i,y'} & X_{i,y''} \\ X'_{j,y} & X'_{j,y'} & X'_{j,y''} \\ X''_{K,y} & X''_{K,y'} & X''_{K,y''} \end{vmatrix}}{\begin{vmatrix} X'_{j,y'} & X'_{j,y''} \\ X''_{K,y'} & X''_{K,y''} \end{vmatrix}} \quad \ldots \quad \ldots \quad (14)$$

This expresses all the partial derivatives of third rank in terms of partial derivatives of first rank and consequently in terms of the 9 basic partial derivatives.

4. NECESSARY ADJUSTMENTS IN SUBSIDIARY LOADING CONDITIONS

When X'_j, X''_K, z are kept constant, we have from (12) and (13)

$$X'_{j,y}\frac{\delta y}{y} + X'_{j,y'}\frac{\delta y'}{y'} + X'_{j,y''}\frac{\delta y''}{y''} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

$$X''_{K,y}\frac{\delta y}{y} + X''_{K,y'}\frac{\delta y'}{y'} + X''_{K,y''}\frac{\delta y''}{y''} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

From (15) and (16)

$$\frac{\partial(\log y')}{\partial(\log y)} = \frac{\begin{vmatrix} X'_{j,y''} & X'_{j,y} \\ X''_{K,y''} & X''_{K,y} \end{vmatrix}}{\begin{vmatrix} X'_{j,y'} & X'_{j,y''} \\ X''_{K,y'} & X''_{K,y''} \end{vmatrix}} \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

and

$$\frac{\partial(\log y'')}{\partial(\log y)} = \frac{\begin{vmatrix} X'_{j,y} & X'_{j,y'} \\ X''_{K,y} & X''_{K,y'} \end{vmatrix}}{\begin{vmatrix} X'_{j,y'} & X'_{j,y''} \\ X''_{K,y'} & X''_{K,y''} \end{vmatrix}}, \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

giving the necessary adjustments in terms of partial derivatives of first rank only. For adjustment in case of partial derivatives of second rank the formula is easily seen to be

$$\frac{\partial\,(\log y')}{\partial\,(\log y)} = -\frac{X'_{j,y}}{X'_{j,y'}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

5. UNIQUENESS OF MAXIMUM VALUE OF RATE OF CHANGE OF ENERGY FOR COMPOSITE CHARGES

Since

$$\frac{d}{dt}(pv) = \frac{dp}{dt}v + p\frac{dv}{dt}$$

$$= v\frac{dp}{dt} + \frac{Ap^2}{w_1}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

and since at the end of each stage of burning p, v are continuous but $\frac{dp}{dt}$ can become discontinuous, it follows that $\frac{d}{dt}(pv)$ can also receive an increment at the end of a stage. It has, however, been shown (Kapur, 1956) that at the end of the rth stage the increment in $\frac{dp}{dt}$ is

$$-\frac{p_r F_r C_r \beta_r (1-\theta_r)}{D_r(x_r+l)}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

where F_r, C_r, β_r, D_r, θ_r are the force-constant, charge mass, rate of burning constant, ballistic size and form-factor of the rth component charge (the charges being so numbered that $\frac{\beta_1}{D_1} \geqslant \frac{\beta_2}{D_2} \geqslant \ldots \geqslant \frac{\beta_n}{D_n}$, and p_r, x_r are the pressure and shot-travel till the end of the rth stage. Accordingly increment in $\frac{d}{dt}(pv)$ at the end of the rth stage is

$$-\frac{p_r v_r F_r C_r \beta_r (1-\theta_r)}{D_r(x_r+l)}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

where v_r is the velocity at the end of the rth stage. Since this increment is essentially non-positive, it follows that at the end of a stage $\frac{d}{dt}(pv)$ can change from positive to negative but not vice versa. Again using (14)–(18) of Kapur (1956)

$$\frac{d}{dt}(pv) = \frac{pv}{x+l}[K_r(a_r-b_r) - v(2K_r+1)] + \frac{Ap_r^2}{W_1}$$

$$= \frac{p}{x+l}\left\{\begin{array}{c} vK_r(a_r-b_r) - v^2(2K_r+1) \\ +K_r(a_r-v)(b_r+v) \end{array}\right\}$$

$$= \frac{p}{x+l}[K_r a_r b_r + 2vK_r(a_r-b_r) - v^2(3K_r+1)], \quad \ldots \quad (23)$$

where

$$K_r = \frac{W_1}{A^2}\sum_{i=r}^{n} F_i C_i \beta_i'^2\, \theta_i + \tfrac{1}{2}(\gamma-1) \quad \ldots \quad \ldots \quad \ldots \quad (24a)$$

$$K_r a_r b_r = \frac{\sum_{i=1}^{r-1} F_i C_i + \sum_{i=r}^{n} F_i C_i z_{i_0}}{W_1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (24b)$$

$$K_r(a_r-b_r) = \frac{1}{A}\sum^{n} F_i C_i \beta_i'(1-\theta_i+2\theta_i f_{i_0}) \quad \ldots \quad \ldots \quad \ldots \quad (24c)$$

$K_r a_r b_r$, $K_r(a_r - b_r)$ are essentially non-negative and K_r will also in general be non-negative. If, however, $3K_r+1$ is $-ve$, then $\frac{t}{dt}(pv)$ is positive throughout the rth stage and there can be no maximum in this interval. If $3K_r+1$ is $+ve$, then

$$\frac{d}{dt}(pv) = \frac{p}{x+l}(3K_r+1) \times [(\alpha_r+v)(\beta_r-v)]. \quad .. \quad .. \quad (25)$$

where α_r, β_r are positive.

Since v increases in the rth stage, $\frac{d}{dt}(pv)$ can change from positive to negative in the rth stage, but not vice versa.

Thus we find that both during a stage as well as at the end of the stage $\frac{d}{dt}(pv)$ can change from positive to negative only and thus once $\frac{d}{dt}(pv)$ has become negative during a stage or at the end of a stage, it cannot become positive again. This establishes the uniqueness of maximum pv.

For the maximum pv to occur during the rth stage

(i) $3K_r+1 > 0$

(ii) If β_r be the $+ve$ root of

$$K_r a_r b_r + 2K_r(a_r-b_r)v - v^2(3K_r+1) = 0, \quad .. \quad .. \quad (26)$$

then

$$\beta_r - v_{r-1} > 0 \text{ and } \beta_r - v_r < 0 \quad .. \quad .. \quad .. \quad (27)$$

The maximum pv will occur at the end of the rth stage of

$$\text{(i) } \beta_r - v_r > 0 \text{ and } \beta_{r+1} - v_r < 0 \quad .. \quad .. \quad .. \quad (28a)$$

or

$$\text{(ii) } \beta_r - v_r = 0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (28b)$$

or

$$\text{(iii) } \beta_{r+1} - v_r = 0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (28c)$$

After all-burnt

$$\sum_{i=1}^{n} F_r C_i = Ap(x+l) + \tfrac{1}{2}(\gamma-1)w_1 v^2, \quad .. \quad .. \quad .. \quad (29)$$

and

$$p(x+l)^{\gamma} = p_B(x_B+l)^{\gamma}, \quad .. \quad .. \quad .. \quad .. \quad (30)$$

so that

$$\frac{dp}{dt} = -\frac{\gamma p v}{x+l}$$

and

$$\frac{d}{dt}(pv) = \frac{p}{(x+l)}\left[-\gamma v^2 + \frac{Ap(x+l)}{W_1}\right] \quad .. \quad .. \quad .. \quad (31)$$

$$= \frac{p}{(\gamma-1)(x+l)}\frac{1}{W_1}\left[-2\gamma \sum_{i=1}^{n} F_r C_i + (3\gamma-1)Ap(x+l)\right]$$

$$= \frac{p}{(\gamma-1)W_1}\frac{1}{x+l}\left[-2\gamma \sum_{i=1}^{n} F_r C_i + \frac{(3\gamma-1)Ap_B(x_B+l)^{\gamma}}{(x+l)^{\gamma-1}}\right] \quad .. \quad (32)$$

where suffix B corresponds to all-burnt.

The maximum pv will occur after all-burnt if

$$\frac{(3\gamma-1)Ap_B(x_B+l)^{\gamma}}{(x+l)^{\gamma}}-2\gamma\sum_{i=1}^{n}F_rC_i \quad \cdots \quad (33)$$

is positive at $x = x_B$, but becomes negative for some x lying between x_B and x_3. If it is negative or zero at all-burnt, maximum value will occur there. If it is non-negative at the muzzle, maximum will occur there.

6. The Equation of Pressure-time Curve

The dynamical equation of internal ballistics is

$$W_1\frac{dv}{dt} = Ap,$$

which becomes

$$\frac{d\eta}{d\tau} = \zeta, \quad \cdots \quad (34)$$

in terms of the dimensionless variables

$$\eta = \frac{AD}{FC\beta}v \quad \cdots \quad (35a)$$

$$\zeta = \frac{Al}{FC}p \quad \cdots \quad (35b)$$

$$\tau = \frac{AD}{W_1\beta l}t \quad \cdots \quad (35c)$$

From Kapur (1959*c*)

$$\zeta = \frac{ab\theta'}{M}\left(1+\frac{\eta}{a}\right)^{1+\frac{a}{\theta'(a+b)}}\left(1-\frac{\eta}{b}\right)^{1+\frac{b}{\theta'(a+b)}}, \quad \cdots \quad (36)$$

where

$$\theta' = \frac{\theta}{M}+\tfrac{1}{2}(\gamma-1),\ \theta'ab = Mz_0,\ \theta'(b-a) = 1-\theta+2rf_0 \quad \cdots \quad (37)$$

$$\therefore \quad \tau = \int_0^{\eta}\frac{d\eta}{\frac{ab\theta'}{M}\left(1+\frac{\eta}{a}\right)^{1+\frac{a}{\theta'(a+b)}}\left(1-\frac{\eta}{b}\right)^{1+\frac{b}{\theta'(a+b)}}} \quad \cdots \quad (38)$$

Putting

$$b-\eta = (a+b)u, \quad \cdots \quad (39)$$

$$\tau = -\int_{\frac{a}{a+b}}^{u}\frac{(a+b)^{-1-\frac{a}{\theta'(a+b)}-\frac{b}{\theta'(a+b)}}\,a^{1+\frac{a}{\theta'(a+b)}}\,b^{1+\frac{b}{\theta'(a+b)}}}{u^{1+\frac{b}{\theta'(a+b)}}(1-u)^{1+\frac{a}{\theta'(a+b)}}}du$$

$$= K\int_{u}^{\frac{a}{a+b}}u^{-1-\frac{b}{\theta'(a+b)}}(1-u)^{-1-\frac{a}{\theta'(a+b)}}du \quad \cdots \quad (40)$$

where K is a constant.

By using reduction formulae for connecting $\int x^m(1-x)^n\,dx$ with $\int x^{m+2}(1-x)^n\,dx$ and $\int x^m(1-x)^{n+2}\,dx$, we can increase the exponents of u and $(1-u)$ in the above integral, till they become greater than -1 and hence the time-velocity relation can be expressed in terms of incomplete Beta functions. This together with pressure-velocity relation gives the parametric equations for the pressure-time relation.

If the shot-start pressure is zero

$$\zeta = \frac{\eta(1+\theta)}{M}\left[1-\frac{\eta\theta'}{1+\theta}\right]^{1+\frac{1}{\theta'}}$$

$$\tau = \frac{M}{1+\theta}\int_0^{\eta} \frac{d\eta}{\eta\left(1-\frac{\eta\theta'}{1+\theta}\right)^{1+\frac{1}{\theta'}}} \quad \ldots \quad \ldots \quad \ldots \quad (41)$$

The integral would diverge and to avoid this we measure time backwards from the instant of all-burnt. If $1+\frac{1}{\theta'}$ is an integer, the integral can be easily integrated in finite terms, as in Tawakley (1959) where he has studied the particular case $\theta = 0$, $\gamma = 1{\cdot}25$, $1+\frac{1}{\theta'} = 9$. Even if $1+\frac{1}{\theta'}$ is a rational number of the form $\frac{p}{q}$, the integral can be evaluated in finite terms, the integration being particularly simple if $q = 2$ or 3.

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