LINEARIZED MOTION OF A SPHERE ALONG THE AXIS OF A ROTATING LIQUID

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Abstract

Taylor's problem of the motion of a sphere along the axis of a rotating liquid has been solved in this paper by taking the hydrodynamical flow equations in linearized form. The flow due to a source on the axis of rotation has first been obtained whence through calculation of the flow due to a doublet the motion due to a moving sphere has been deduced. The point to be noticed in this solution is that with linearization the indeterminacy observed by Taylor, taking the non-linear equations into consideration, is removed and completely determinate result consistent with that of Taylor is obtainable.

I. Introduction

The chief hydrodynamical interest in the motion of a sphere along the axis of a uniformly rotating liquid arises from the unusual nature of the flow. The following introduction will supply the background of a solution to the problem discussed in this paper.

The first experiment on such motion in rotating liquids was performed by Taylor (1922) who found that at small values of $ka = 2\Omega a/U$, Ω being the angular velocity of the liquid, a the radius of the sphere, and U the velocity of the sphere, the disturbance in the rotating liquid was the same as that for the case of a sphere moving uniformly along the axis when $\Omega = 0$. Taylor found that for $ka \approx 2\pi$, a column of liquid, of the same diameter as the sphere, was apparently pushed along in front of the sphere. In 1953 Long carried out an experiment with water in a cylindrical vessel, using a body with a spherical nose and a conical tail which was made to move along the axis of the cylinder. When $R_0 = 1/(kb)$ was less than about 0·03, 'b' being the inside radius of the vessel, Long found as in Taylor's observation that the liquid in the central part of the enveloping cylinder C, circumscribing the body was pushed ahead of the body, but the liquid near the boundary passed round to the rear of the body and the liquid to the rear of the body and inside C was not carried along with it; waves of small amplitude and very short wavelength were seen on the downstream side of the body. For higher values of R_0, waves on the downstream side of the body became longer, but no wave motion appeared on the upstream side.

In 1922 Taylor undertook theoretical investigation to see if the phenomena observed by him could be explained by the hydrodynamical equations of motion. His results can be seen as follows. Let (r, θ, z) be cylindrical polar co-ordinates of a point with origin at the centre of the sphere and the axis of rotation as $0z$, and let (v, w, u) be the corresponding components of velocity. We suppose that the sphere is at rest and that in addition to the rotation the liquid streams past it with uniform velocity $u = -U$ at $z = +\infty$. From the equation of continuity a stream function ψ may be introduced such that

$$u = +\frac{1}{r}\frac{\partial \psi}{\partial r}, \quad v = -\frac{1}{r}\frac{\partial \psi}{\partial z} \quad \ldots \quad (1)$$

Long (1953) derived the hydrodynamical equations of motion in terms of ψ as follows

$$\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+k^2\right)\psi=-\frac{1}{2}Uk^2r^2 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

while the appropriate boundary conditions were

$$\psi\to-\tfrac{1}{2}Ur^2 \quad \text{as} \quad z\to+\infty, \quad \text{and} \quad \psi=0 \quad \text{on the sphere.} \quad \ldots \quad (3)$$

In 1922 Taylor adopting spherical polar co-ordinates (R, θ, ϕ) wrote down the same equations in polar form. The solution which satisfied his equation and the corresponding boundary conditions was of the form

$$\psi=-\frac{U}{2}R^2\sin^2\theta+D\sin^2\theta\left[\cos(kR+\epsilon)-\frac{\sin(kR+\epsilon)}{kR}\right], \quad \ldots \quad (4)$$

where the two arbitrary constants D and ϵ of his solution satisfied the single equation

$$D\left[\cos(ka+\epsilon)-\frac{\sin(ka+\epsilon)}{ka}\right]=\frac{1}{2}Ua^2. \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

All the necessary boundary conditions being satisfied Taylor was thus left with an indeterminate problem. Long (1953) showed that nearly every solution of (2) satisfied the boundary condition as $z\to+\infty$, so that another boundary condition was needed to complete the solution.

In 1956 Fraenkel investigated several patterns of flow of liquid through the contraction and expansion of a pipe or past bodies of revolution on the pipe axis. The flow far upstream was assumed to have a constant axial velocity U and constant angular velocity Ω about the pipe axis. In particular Fraenkel obtained a solution to the flow of an unbounded liquid past a sphere on the axis of flow when $ka = 2\Omega a/U$ is small by letting the radius of the pipe $b\to\infty$, a being the radius of the sphere. Making use of Fourier transform Fraenkel solved the differential equation (2) with the boundary condition for a point source on the axis of the pipe of radius b, and also for a doublet which latter is equivalent to that for a sphere. He further made the special assumption that there would be no wave-like motion on the upstream side. After solving this problem for motion within a pipe of finite radius b, he obtained the result for motion when the liquid is unbounded by simply proceeding to the limit $b\to\infty$. The validity of the limiting process under the special assumption, tacitly assumed by Fraenkel, was questioned by Stewartson (1958).

In 1958 Stewartson obtained solution of (2) satisfying the boundary condition at infinity. The second boundary condition on the sphere (3) gave an infinite set of linear equations; a numerical evaluation of the constants of which showed that for small value of $ka \leqslant 2$ all the coefficients except the first were small; but for larger ka they tended to assume very large values.

The above theoretical discussions by Taylor, Long, Fraenkel and Stewartson were all carried out on the basis of the accurate (non-linear) Euler equations applicable to inviscid liquids. Stewartson (1952) studied the case of linearized slow motion when the sphere is started with an initial impulse. His solution showed some interesting observable features of such motion but failed to represent a motion continuous throughout the liquid.

In section 3 of this paper a solution due to a source of strength m placed on the axis of rotation of an infinite mass of inviscid liquid which has a constant axial velocity far upstream has been obtained on the *basis of a linearized theory*. It is assumed that all disturbances vanish on the far upstream side. In section 4 a solution due to a doublet which is equivalent to that due to a sphere placed on the axis of rotation has been derived from the solution due to a source. This problem is similar to that of Taylor except for linearization. The Stokes stream function for the resultant flow has been deduced and compared with the corresponding stream function (4) obtained by Taylor by using the strict non-linear equations. It has been shown that our result is in agreement with that of Taylor if we only put $\epsilon = 0$ in his result (4) given above. In our solution of the linearized problem we have formally satisfied all the conditions of this problem and it turns out that linearization which is so to say a first approximation calculation of the real problem dispenses with the indeterminacy in Taylor's solution (4) of the real problem. A whole set of downstream waves does not appear in our solution. But it should be noticed that our solution of the linearized problem is valid everywhere *except on the z-axis.* It seems that linearization in the present case introduces unreality and a solution continuous in the whole domain does not appear to be possible.

2. Formulation of the Problem

Let us suppose that a liquid unlimited in all directions is rotating about the axis of z with uniform angular velocity Ω, and that the liquid has a constant axial velocity $-U$ far upstream. The perturbation of the general velocity of the liquid due to a source of strength m at the origin is supposed to be sufficiently small for the squares and products of perturbation velocities to be neglected.

The momentum equation for steady incompressible inviscid flow is

$$\left(\vec{q}\cdot\vec{\nabla}\right)\vec{q} = -\frac{1}{\rho}\vec{\nabla}p, \qquad (6)$$

where $\vec{q}$ is the liquid velocity, ρ the density and p the pressure. To solve the present problem we shall use cylindrical co-ordinates (r, θ, z) where r, θ are polar co-ordinates in a plane normal to the z-axis (the axis of rotation of the liquid), and let the components of liquid velocity be v, $w = \Omega r + w_1$, $u = -U + u_1$ along the directions of r, θ, z respectively. We shall assume v, w_1 and u_1 to be small so that we may neglect the squares and products of small quantities in the equations of motion (6). The equations of motion then reduce to

$$U\frac{\partial v}{\partial z} + \frac{1}{r}(\Omega^2 r^2 + 2\Omega r w_1) = \frac{\partial}{\partial r}(p/\rho) \qquad (7)$$

$$U\frac{\partial u_1}{\partial z} = \frac{\partial}{\partial z}(p/\rho) \qquad (8)$$

$$-U\frac{\partial w_1}{\partial z} + 2\Omega v = 0 \qquad (9)$$

since the motion being symmetrical about the axis of rotation is independent of θ. These equations are to be satisfied together with the equation of continuity and the boundary conditions. Now, to obtain the solution when there is a source of strength m at the origin (from which other solutions can be built up) we write the equation of continuity as

$$\frac{\partial u_1}{\partial z} + \frac{v}{r} + \frac{\partial v}{\partial r} = \frac{m}{2\pi r}\delta(z)\,\delta(r), \qquad (10)$$

where $\delta(z)$, $\delta(r)$ are the delta functions of Dirac, or '*impulse function*'. The right hand side of (10) vanishes except at the origin, but its integral over any volume including the origin is m. Similar terms do not, however, appear on the right hand side of (7), (8) or (9) because the source is taken to be a source of mass but not of momentum. For the boundary conditions we shall assume that all *disturbance velocities* vanish far upstream (at $z = +\infty$).

To obtain an equation for the single variable w_1, we proceed as follows: From (9) we have

$$v = \frac{U}{2\Omega}\frac{\partial w_1}{\partial z} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

From (10) we have

$$\frac{\partial u_1}{\partial z} = \frac{m}{2\pi r}\,\delta(z)\,\delta(r) - \frac{1}{r}\frac{\partial}{\partial r}(rv)$$

$$= \frac{m}{2\pi r}\,\delta(z)\,\delta(r) - \frac{U}{2\Omega}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_1}{\partial z}\right) \qquad \ldots \quad \ldots \quad (12)$$

Substituting (11) and (12) in (7) and (8) we have

$$\frac{U^2}{2\Omega}\frac{\partial^2 w_1}{\partial z^2} + \Omega^2 r + 2\Omega w_1 = \frac{\partial}{\partial r}(p/\rho) \qquad \ldots \quad \ldots \quad \ldots \quad (13)$$

$$\frac{mU}{2\pi r}\,\delta(z)\,\delta(r) - \frac{U^2}{2\Omega}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_1}{\partial z}\right) = \frac{\partial}{\partial z}(p/\rho) \qquad \ldots \quad \ldots \quad (14)$$

Eliminating p/ρ from (13) and (14) we have the equation for w_1,

$$\left[\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \left(k^2 - \frac{1}{r^2}\right)\right]\frac{\partial w_1}{\partial z} = \frac{mk}{2\pi}\,\delta(z)\,\frac{d}{dr}\left(\frac{\delta(r)}{r}\right)$$

$$= \frac{mk}{2\pi}\,\delta(z)\left[\frac{\delta'(r)}{r} - \frac{\delta(r)}{r^2}\right]$$

$$= -\frac{mk}{\pi r^2}\,\delta(z)\delta(r), \qquad \ldots \quad \ldots \quad (15)$$

where we have used the relation

$$f(r)\delta'(r) = f(0)\delta'(r) - f'(0)\delta(r)$$

and put

$$k = 2\Omega/U \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

We now write w_1 as a Fourier integral

$$w_1(r, z) = \int_{-\infty}^{+\infty} e^{i\alpha z} w_0(r, \alpha)\,d\alpha \qquad \ldots \quad \ldots \quad \ldots \quad (17)$$

Then by substitution in (15), we have

$$\int_{-\infty}^{+\infty} i\alpha\left[\left(-\alpha^2 + k^2 - \frac{1}{r^2}\right)w_0 + w_0'' + \frac{1}{r}w_0'\right]e^{i\alpha z}\,d\alpha = -\frac{mk}{\pi r^2}\,\delta(r)\delta(z)$$

$$= -\frac{mk}{\pi r^2}\,\delta(r)\,\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\alpha z}\,d\alpha$$

Or, we have

$$i\alpha\left[w_0'' + \frac{1}{r}w_0' + w_0\left(k^2 - \alpha^2 - \frac{1}{r^2}\right)\right] = -\frac{mk}{2\pi^2 r^2}\,\delta(r), \qquad \ldots \quad \ldots \quad (18)$$

where dashes denote differentiation with respect to r.

This w_0 is to be obtained from (18), which states that for $r > 0$ the simple equation

$$r^2 w_0'' + r w_0' + w_0\{-1 + r^2(k^2 - \alpha^2)\} = 0 \quad \ldots \quad (19)$$

is satisfied by w_0; but w_0, w_0' are discontinuous at $r = 0$, in such a way that

$$\lim_{r \to +0} \left[r^2 w_0' - r w_0 + \int w_0 dr \right] - 0 = -\frac{mk}{2\pi^2 i\alpha} \quad \ldots \quad (20)$$

The second term on the left hand side of (20) is the value of the bracketed expression at $r = 0$ and is equal to zero because on the axis of rotation $w = 0$ i.e. $w_1 = 0$ and consequently $w_0 = 0$.

3. Solution of the Problem

The solution of the differential equation (19) satisfying the boundary condition $w_0 \to 0$ as $r \to \infty$ is given by

$$\left.\begin{aligned} w_0 &= A J_1[r\sqrt{k^2-\alpha^2}] + B Y_1[r\sqrt{k^2-\alpha^2}] \text{ for } \alpha < k \\ &= C K_1[r\sqrt{\alpha^2 - k^2}] \qquad \text{for } \alpha > k \end{aligned}\right\}, \quad \ldots \quad (21)$$

where J_1, Y_1 are the Bessel function of the first and second kind of order one and K_1 is the modified Bessel function of the second kind of order one.

To satisfy the relation (20) we must have

$$B = C = 0 \text{ and } A = \frac{mk}{2\pi^2 i\alpha}\sqrt{k^2-\alpha^2} \quad \ldots \quad (22)$$

because when (21) is substituted in (20) the coefficients of B and C become infinite as $r \to +0$. Therefore (21) becomes

$$\left.\begin{aligned} w_0 &= \frac{mk}{2\pi^2 i\alpha}\sqrt{k^2-\alpha^2}\, J_1[r\sqrt{k^2-\alpha^2}] \text{ for } \alpha < k \\ &= 0 \qquad \text{for } \alpha > k \end{aligned}\right\} \quad \ldots \quad (23)$$

From (17) we obtain

$$\frac{\partial w_1}{\partial z} = \int_{-\infty}^{+\infty} i\alpha e^{i\alpha z} w_0 d\alpha = \frac{mk}{2\pi^2}\int_{-k}^{+k} e^{i\alpha z}\sqrt{k^2-\alpha^2} J_1[r\sqrt{k^2-\alpha^2}]\, d\alpha \quad \text{by (23)}$$

$$= \frac{mk}{\pi^2}\int_0^k \cos(\alpha z)\sqrt{k^2-\alpha^2}\, J_1[r\sqrt{k^2-\alpha^2}]\, d\alpha$$

$$= -\frac{mkr}{\pi^2}\left[\frac{k(r^2+z^2)^{1/2}\cos[k(r^2+z^2)^{1/2}] - \sin[k(r^2+z^2)^{1/2}]}{(r^2+z^2)^{3/2}}\right] \quad \ldots \quad (24)$$

Integrating (24) with respect to z we have

$$w_1 = -\frac{mkr}{\pi^2}\int^z \frac{k(r^2+z^2)^{1/2}\cos[k(r^2+z^2)^{1/2}] - \sin[k(r^2+z^2)^{1/2}]}{(r^2+z^2)^{3/2}}\, dz + F(r)$$

Now since according to our assumption $w_1 \to 0$ as $z \to +\infty$, we have

$$w_1 = \frac{mkr}{\pi^2} \int_z^{+\infty} \frac{k(r^2+z^2)^{1/2} \cos [k(r^2+z^2)^{1/2}] - \sin [k(r^2+z^2)^{1/2}]}{(r^2+z^2)^{3/2}} dz \quad .. \quad (25)$$

From (11)

$$v = \frac{1}{k} \frac{\partial w_1}{\partial z} = -\frac{mr}{\pi^2} \left[\frac{k(r^2+z^2)^{1/2} \cos [k(r^2+z^2)^{1/2}] - \sin [k(r^2+z^2)^{1/2}]}{(r^2+z^2)^{3/2}} \right] \quad (26)$$

From (12) and (24) we have

$$\frac{\partial u_1}{\partial z} = \frac{m}{2\pi r} \delta(z)\delta(r) - \frac{1}{kr} \frac{\partial}{\partial r} \left(r \frac{\partial w_1}{\partial z} \right) \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (27)$$

$$= \frac{m}{\pi^2} \left[2k \frac{\cos [k(r^2+z^2)^{1/2}]}{(r^2+z^2)} - 2 \frac{\sin [k(r^2+z^2)^{1/2}]}{(r^2+z^2)^{3/2}} - k^2 r^2 \frac{\sin [k(r^2+z^2)^{1/2}]}{(r^2+z^2)^{3/2}} \right.$$
$$\left. - 3kr^2 \frac{\cos [k(r^2+z^2)^{1/2}]}{(r^2+z^2)^2} + 3r^2 \frac{\sin [k(r^2+z^2)^{1/2}]}{(r^2+z^2)^{5/2}} \right] \text{ for } r > 0. \quad .. \quad (28)$$

Integrating (27) with respect to z we have

$$u_1 = \frac{m}{2\pi r} H(z)\delta(r) - \frac{1}{kr} \frac{\partial}{\partial r} (rw_1) + f(r),$$

where $H(z)$ is the Heaviside unit function. Substituting the value of w_1 from (25) in the above equation we have for $r > 0$, or integrating (28) with respect to z we have

$$u_1 = \frac{m}{\pi^2} \left[-k^2 \int^z \frac{\sin [k(r^2+z^2)^{1/2}]}{(r^2+z^2)^{1/2}} dz - kz \frac{\cos [k(r^2+z^2)^{1/2}]}{(r^2+z^2)} + z \frac{\sin [k(r^2+z^2)^{1/2}]}{(r^2+z^2)^{3/2}} \right] + f(r).$$

As $z \to +\infty$, $u_1 \to 0$, we must have

$$u_1 = \frac{m}{\pi^2} \left[k^2 \int_z^{+\infty} \frac{\sin [k(r^2+z^2)^{1/2}]}{(r^2+z^2)^{1/2}} dz - kz \frac{\cos [k(r^2+z^2)^{1/2}]}{(r^2+z^2)} + z \frac{\sin [k(r^2+z^2)^{1/2}]}{(r^2+z^2)^{3/2}} \right] \quad (29)$$

For $r > 0$, the equation of continuity (10) suggests that we may express the velocity components u_1, v by Stokes stream function $\psi_1(r, z)$ as follows:

$$u_1 = \frac{1}{r} \frac{\partial \psi_1}{\partial r}, \quad v = -\frac{1}{r} \frac{\partial \psi_1}{\partial z} \quad .. \quad .. \quad .. \quad (30)$$

From (30) and (11) we have

$$v = -\frac{1}{r} \frac{\partial \psi_1}{\partial z} = \frac{1}{k} \frac{\partial w_1}{\partial z}$$

or

$$\psi_1 = -\frac{r}{k} w_1 + g(r) = g(r) - \frac{mr^2}{\pi^2} \int_z^{+\infty} \frac{k(r^2+z^2)^{1/2} \cos [k(r^2+z^2)^{1/2}] - \sin [k(r^2+z^2)^{1/2}]}{(r^2+z^2)^{3/2}} dz$$

when $z \longrightarrow +\infty$, $\psi_1 \to 0$. Therefore $g(r) = 0$.

Hence

$$\psi_1 = -\frac{mr^2}{\pi^2} \int_z^{+\infty} \frac{k(r^2+z^2)^{1/2} \cos [k(r^2+z^2)^{1/2}] - \sin [k(r^2+z^2)^{1/2}]}{(r^2+z^2)^{3/2}} dz \quad (31)$$

4. Solution of the problem for a doublet placed on the axis at the origin

The disturbance stream function ψ_2 due to a doublet of strength μ placed on the axis of rotation can be obtained from ψ_1 which is the corresponding stream function due to a source, by differentiating the above result with respect to z and replacing m by μ. We have then

$$\psi_2 = \frac{\mu r^2}{\pi^2} \frac{k(r^2+z^2)^{1/2} \cos [k(r^2+z^2)^{1/2}] - \sin [k(r^2+z^2)^{1/2}]}{(r^2+z^2)^{3/2}} \quad .. \quad (32)$$

The expression (32) becomes when expressed in terms of spherical polar co-ordinates (R, θ, ϕ) (i.e. on putting $r = R \sin \theta$, $z = R \cos \theta$)

$$\left.\begin{aligned} \psi_2 &= \frac{\mu}{\pi^2} \sin^2 \theta \left[k \cos (kR) - \frac{\sin (kR)}{R} \right] \\ &= \frac{\mu k}{\pi^2} \sin^2 \theta \left[\cos (kR) - \frac{\sin (kR)}{kR} \right] \end{aligned}\right\} \quad .. \quad .. \quad (33)$$

Now a disturbance stream function of the form $\mu \sin^2\theta f(R)$ leads to the flow about a sphere, since μ may be chosen to make $\Psi = -\frac{U}{2} R^2 \sin^2\theta + \psi_2 = 0$ on $R =$ constant. Moreover we see that if we put $\epsilon = 0$ in Taylor's result (4), the disturbance stream function is in agreement with (33).

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