LOW-TEMPERATURE BEHAVIOUR OF A BOSE GAS WITH HARD-SPHERE INTERACTION

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ABSTRACT

Making use of the energy-momentum relation derived by Brueckner and Sawada, the low-temperature thermodynamical properties of a Bose gas with hard-sphere interaction are investigated. The two cases, (i) when this energy spectrum is similar to that observed in liquid helium II and (ii) when it becomes similar to that derived by Lee, Huang and Yang, are investigated separately. In the first case (where the spectrum exhibits a non-monotonic character) the analytical derivation of the expressions for the various thermodynamical properties is rendered possible by dividing the whole momentum range into three suitable intervals such that within each interval the ϵ-p relation can be approximated to a simple algebraic one. The numerical magnitudes of the various parameters characterizing these algebraic relations are then calculated for the case of a gas with density equal to that of liquid helium II and for three plausible values of the hard-sphere diameter. The calculated results agree qualitatively with those obtained from the experimental study of this liquid. Further, for the second case, we first derive the expressions for the various macroscopic properties of the assembly in terms of suitable mathematical functions and then write down, in the form of series expansions, the two particular cases of interest.

I. INTRODUCTION

The present paper deals with a theoretical investigation of the low-temperature thermodynamical properties of a hard-sphere Bose gas on the basis of the energy-momentum relation derived by Brueckner and Sawada (1957). This relation exhibits, for a certain range of the particle density, a characteristic non-monotonic behaviour similar to that of the elementary excitations in liquid helium II. For low densities, on the other hand, it becomes similar to the monotonic spectrum obtained by Lee, Huang and Yang (1957). We have investigated these two cases separately.

In the following section we employ the non-monotonic energy spectrum for calculating the various macroscopic properties of the assembly. Here, in order to simplify the analytical derivation of the expressions for these various properties, we divide the whole momentum range into three intervals such that within each interval the ϵ-p relation can be approximated to a simple algebraic relation. The final expression for any thermodynamical property of the assembly will then be obtained by adding up the corresponding expressions for these intervals. Next, we write down the expressions for the parameters characterizing the energy-momentum relations in these intervals in terms of the particle density ρ and the hard-sphere diameter a. We also give in a tabulated form the numerical magnitudes of these parameters and their derivatives with respect to density for the case of a gas whose density is equal to that of liquid helium II and for three plausible values of a. The corresponding values obtained from the experimental study of this liquid by Bendt *et al.* (1959) and Atkins (1955) are also listed for the sake of comparison.

In the next section we study the problem in the case when the energy spectrum is effectively monotonic. Firstly, we set up exact expressions for the various thermodynamical quantities in terms of certain suitable mathematical functions

and then write down explicitly, in the form of series expansions, the two limiting cases of interest.

II

The energy spectrum for the elementary excitations in a Bose gas with hard-sphere interaction, as derived by Brueckner and Sawada, is of the form

$$\epsilon = \frac{p}{2m}\left[p^2+2\lambda^2\frac{\hbar^2}{a^2}\frac{\sin\left(\frac{pa}{\hbar}\right)}{\frac{pa}{\hbar}}\right]^{\frac{1}{2}}, \quad .. \quad .. \quad .. \quad (1)$$

where m is the particle mass and λ^2 (which is a function of the particle density ρ and the hard-sphere diameter a) is given by the relation

$$4\pi^2\rho a^3 = \lambda^2\int_0^\infty \frac{\sin^2 x\,dx}{x^2+\lambda^2\frac{\sin x}{x}}. \quad .. \quad .. \quad .. \quad (2)$$

It is readily seen that equation (1) exhibits a minimum at $p = p_0$ given by the relation

$$2\theta_0{}^3+\lambda^2(\sin\theta_0+\theta_0\cos\theta_0) = 0, \quad .. \quad .. \quad .. \quad (3)$$

where θ_0 stands for $\frac{p_0 a}{\hbar}$. The energy Δ at the minimum is

$$\Delta = \frac{\hbar^2}{2ma^2}[\theta_0{}^4+2\lambda^2\theta_0\sin\theta_0]^{\frac{1}{2}}. \quad .. \quad .. \quad .. \quad (4)$$

It may be noticed that equation (3) does not possess a solution for λ^2 less than the critical value 18·93 . . . , which is obtained by solving simultaneously this equation and its derivative with respect to θ_0. (The value of ρa^3 corresponding to this value of λ^2 is about 0·11). Further when λ^2 exceeds the value 42·10 . . . the energy at the minimum does not remain real. This critical value of λ^2 is obtained as a solution of the simultaneous equations (3) and (4) with $\Delta = 0$. The corresponding value of ρa^3 is about 0·25. Thus it is within these two limits of the particle density that the spectrum (1) exhibits a non-monotonic character which is typical of the elementary excitations in liquid helium II. For densities less than the lower limit the energy-momentum relation (1) becomes effectively monotonic and for densities larger than the upper limit it yields unphysical results. We, therefore, firstly investigate this spectrum within the above-mentioned limits.

Because of the typical non-monotonic form of the spectrum an analytical evaluation of the integrals appearing as such in the expressions for the various thermodynamical quantities would be quite involved. This difficulty, however, can be met with by dividing the energy spectrum into regions such that within each region the ϵ-p relation can be approximated to a simple algebraic one. It may be mentioned here that this process does not introduce any significant error into the final results†. For all practical purposes one may consider the following intervals of momentum as constituting the three separate regions:

1. $0 < p < p_1$, with the linear relation

$$\epsilon = pc, \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (5)$$

† In a separate investigation (to be published) we have employed a similar process in order to calculate the various thermodynamical properties of liquid helium II by using the experimentally observed energy spectrum obtained in reference 2. Our results are in exact agreement with those computed by Bendt *et al.* through numerical integration.

2. $p_1 < p < p_2$, with the parabolic relation

$$\epsilon = \Delta^* - (p-p_0^*)^2/2\mu^*, \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

3. $p_2 < p < \infty$, with the relation

$$\epsilon = \Delta + (p-p_0)^2/2\mu. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

The identification of the region 1 with 'phonons' and the region 3 with 'rotons' is well known.

Let us now determine the different parameters characterizing the relations (5)–(7) in terms of the parameters appearing in the relation (1). For the region 1, we obtain

$$c = \operatorname*{Lt}_{p\to 0} (\epsilon/p)$$

$$= \frac{\lambda}{\sqrt{2}} \frac{\hbar}{ma}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

For the region 2, one may expand the relation (1) in the neighbourhood of its maximum ($p = p_0^*$) and then compare it with (6). This gives

$$\Delta^* = \epsilon_{p=p_0^*} = \frac{\hbar^2}{2ma^2}[\theta_0^{*4} + 2\lambda^2\theta_0^* \sin\theta_0^*]^{\frac{1}{2}}, \quad \ldots \quad \ldots \quad (9)$$

and

$$\frac{\mu^*}{m} = \frac{-1}{m}\left(\frac{\partial^2\epsilon}{\partial p^2}\right)^{-1}_{p=p_0^*} = -\frac{[\theta_0^{*4} + 2\lambda^2\theta_0^* \sin\theta_0^*]^{\frac{1}{2}}}{[6\theta_0^{*2} + \lambda^2(2\cos\theta_0^* - \theta_0^* \sin\theta_0^*)]}, \quad \ldots \quad (10)$$

where $\theta_0^* = p_0^* a/\hbar$ is also given by equation (3).

In a similar way one gets for the region 3,

$$\Delta = \epsilon_{p=p_0} = \frac{\hbar^2}{2m^2 a}[\theta_0^4 + 2\lambda^2\theta_0 \sin\theta_0]^{\frac{1}{2}}, \quad \ldots \quad \ldots \quad (11)$$

and

$$\frac{\mu}{m} = \frac{1}{m}\left(\frac{\partial^2\epsilon}{\partial p^2}\right)^{-1}_{p=p_0} = \frac{[\theta_0^4 + 2\lambda^2\theta_0 \sin\theta_0]^{\frac{1}{2}}}{[6\theta_0^2 + \lambda^2(2\cos\theta_0 - \theta_0 \sin\theta_0)]}. \quad \ldots \quad \ldots \quad (12)$$

We now write down the expressions for the various thermodynamical properties of the assembly under consideration.

For the total number of excitations N, their inertial density ρ_n and entropy S we have

$$N = \frac{4\pi V}{\hbar^3}\int_0^\infty \left[\exp\left(\frac{\epsilon}{kT}\right) - 1\right]^{-1} p^2\, dp, \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

$$\rho_n = \frac{4\pi}{3kT\hbar^3}\int_0^\infty \left[\exp\left(\frac{\epsilon}{kT}\right) - 1\right]^{-2} \exp\left(\frac{\epsilon}{kT}\right) p^4\, dp, \quad \ldots \quad \ldots \quad (14)$$

and

$$S = \frac{4\pi Vk}{\hbar^3}\int_0^\infty \left[\frac{\epsilon}{kT}\left\{\exp\left(\frac{\epsilon}{kT}\right) - 1\right\}^{-1} - ln\left\{1 - \exp\left(-\frac{\epsilon}{kT}\right)\right\}\right] p^2\, dp. \quad (15)$$

All the other properties of interest follow directly from these ones. Now, in order to solve these integrals we may divide the momentum space into regions as outlined

above. Correspondingly each macroscopic property of the assembly would be obtained as the sum of respective contributions. For the region 1, we get (remembering that $\exp(p_1c/kT) \gg 1$)

$$N_1 = \frac{4\pi Vk^3T^3}{h^3c^3}[2\zeta(3) - \exp(-x_1)(x_1^2+2x_1+2)], \quad .. \quad .. \quad .. \quad (13a)$$

$$\rho_{n1} = \frac{4\pi k^4T^4}{3h^3c^5}[24\zeta(4) - \exp(-x_1)(x_1^4+4x_1^3+12x_1^2+24x_1+24)], \quad .. \quad (14a)$$

and

$$S_1 = \frac{4\pi Vk^4T^3}{h^3c^3}[8\zeta(4) - \exp(-x_1)(x_1^3+4x_1^2+8x_1+8)], \quad .. \quad .. \quad (15a)$$

where $x_1 = p_1c/kT$. The limiting momentum p_1 may here be chosen so as to satisfy the relation

$$p_1c = \Delta - (p_1-p_0^*)^2/2\mu^* \quad .. \quad .. \quad .. \quad .. \quad (16)$$

For the region 2 we have preferred to choose p_2 such that

$$\left.\begin{aligned} p_0^* &= (p_1+p_2)/2, \\ \epsilon(p_1) &= \epsilon(p_2) \end{aligned}\right\} \quad .. \quad .. \quad .. \quad .. \quad (17)$$

that is,

This choice keeps the final results comparatively simpler. For the contributions from this region, we get

$$N_2 = \left(\frac{2\mu^*kT}{p_0^{*2}}\right)^{\frac{1}{2}}\left(\frac{p_0^*}{\hbar}\right)^3 \frac{1}{\rho\pi^2} \exp\left(-\frac{\Delta^*}{kT}\right)\left[I_0 + \left(\frac{2\mu^*kT}{p_0^{*2}}\right) I_2\right], \quad .. \quad (13b)$$

$$\rho_{n2} = \left(\frac{2\mu^*kT}{p_0^{*2}}\right)^{\frac{1}{2}}\left(\frac{p_0^*}{\hbar}\right)^5 \frac{\hbar^2}{3kT\pi^2} \exp\left(-\frac{\Delta^*}{kT}\right)$$
$$\times \left[I_0 + 6\left(\frac{2\mu^*kT}{p_0^{*2}}\right) I_2 + \left(\frac{2\mu^*kT}{p_0^{*2}}\right)^2 I_4\right], \quad .. \quad .. \quad .. \quad (14b)$$

and

$$S_2 = \left(\frac{2\mu^*kT}{p_0^{*2}}\right)^{\frac{1}{2}}\left(\frac{p_0^*}{\hbar}\right)^3 \frac{k}{\rho\pi^2} \exp\left(-\frac{\Delta^*}{kT}\right)\frac{\Delta^*}{kT}$$
$$\times \left[\left\{I_0 + \frac{kT}{\Delta^*}(I_0 - I_2)\right\} + \left(\frac{2\mu^*kT}{p_0^{*2}}\right)\left\{I_2 + \frac{kT}{\Delta^*}(I_2 - I_4)\right\}\right], \quad .. \quad (15b)$$

where

$$I_n = \int_0^{x_2} \exp(x^2) \cdot x^n \, dx, \qquad x_2 = (p_2 - p_0^*)(2\mu^*kT)^{-\frac{1}{2}}. \quad .. \quad .. \quad (18)$$

Such integrals can all be reduced down to I_0 by the use of the reduction formula

$$I_n = \tfrac{1}{2}x_2^{n-1} \exp(x_2^2) - \frac{n-1}{2} I_{n-2} \quad .. \quad .. \quad .. \quad (19)$$

and the values of I_0 may be obtained from Mathematical Tables (Fritz Emde, 1948). Next, for the region 3 the various expressions come out to be

$$N_3 = \left(\frac{2\mu kT}{p_0^2}\right)^{\frac{1}{2}}\left(\frac{p_0}{\hbar}\right)^3 \frac{1}{2\rho\pi^2} \exp\left(-\frac{\Delta}{kT}\right)\left[\left(J_0 + \frac{\sqrt{\pi}}{2}\right)\right.$$
$$\left. + 2\left(\frac{2\mu kT}{p_0^2}\right)^{\frac{1}{2}}(-J_1 + \tfrac{1}{2}) + \left(\frac{2\mu kT}{p_0^2}\right)\left(J_2 + \frac{\sqrt{\pi}}{4}\right)\right], \quad .. \quad .. \quad .. \quad (13c)$$

$$\rho_{s3} = \left(\frac{2\mu kT}{p_0^2}\right)^{\frac{1}{2}}\left(\frac{p_0}{\hbar}\right)^5 \frac{\hbar^2}{6kT\pi^2} \exp\left(-\frac{\Delta}{kT}\right)\left[\left(J_0+\frac{\sqrt{\pi}}{2}\right)+4\left(\frac{2\mu kT}{p_0^2}\right)^{\frac{1}{2}}(-J_1+\tfrac{1}{2})\right.$$
$$\left.+6\left(\frac{2\mu kT}{p_0^2}\right)^{1}\left(J_2+\frac{\sqrt{\pi}}{4}\right)+4\left(\frac{2\mu kT}{p_0^2}\right)^{\frac{3}{2}}(-J_3+\tfrac{1}{2})+\left(\frac{2\mu kT}{p_0^2}\right)^{2}\left(J_4+\tfrac{3}{8}\sqrt{\pi}\right)\right], \quad .. \quad (14c)$$

and

$$S_3 = \left(\frac{2\mu kT}{p_0^2}\right)^{\frac{3}{2}}\left(\frac{p_0}{\hbar}\right)^3 \frac{k}{2\pi^2\rho} \exp\left(-\frac{\Delta}{kT}\right)\frac{\Delta}{kT}\left[\left\{\left(J_0+\frac{\sqrt{\pi}}{2}\right)\right.\right.$$
$$\left.+\frac{kT}{\Delta}(J_0+J_2+\tfrac{3}{4}\sqrt{\pi})\right\}+2\left(\frac{2\mu kT}{p_0^2}\right)^{\frac{1}{2}}\left\{(-J_1+\tfrac{1}{2})+\frac{kT}{\Delta}(-J_1-J_3+1)\right\}$$
$$\left.+\left(\frac{2\mu kT}{p_0^2}\right)\left\{\left(J_2+\frac{\sqrt{\pi}}{4}\right)+\frac{kT}{\Delta}(J_2+J_4+\tfrac{5}{8}\sqrt{\pi})\right\}\right], \quad .. \quad .. \quad .. \quad (15c)$$

where

$$J_n = \int_0^{x_3} \exp(-x^2)x^n\,dx, \qquad x_3 = (p_0-p_2)(2\mu kT)^{-\frac{1}{2}}. \quad .. \quad (20)$$

Such integrals may also be reduced down to J_0 by the use of the reduction formula

$$J_n = -\tfrac{1}{2}x_3^{n-1}\exp(-x_3^2)+\frac{n-1}{2}J_{n-2}. \quad .. \quad .. \quad (21)$$

and the values of J_0 may again be obtained from the relevant Tables (Fritz Emde, 1948).

By substituting into the foregoing expressions the numerical values of the various parameters as given by relations (8) to (12), one can study quantitatively the thermodynamical properties of the system. We have listed in Table I the values of these parameters for a hard-sphere gas with $\rho = 2{\cdot}18 \times 10^{22}$ cm.$^{-3}$ (which is equal to the particle density in liquid helium II) for three plausible values of a. The experimental results of Bendt *et al.* obtained by neutron scattering are given there for the sake of comparison. We have also listed in the table the values of the logarithmic derivatives of the parameters c, p_0, Δ and μ with respect to density ρ.† These are obtained from the following relations

$$\frac{\rho}{c}\frac{\partial c}{\partial \rho} = \tfrac{1}{2}\frac{\rho a^3}{\lambda^2}\frac{\partial \lambda^2}{\partial(\rho a^3)}, \quad .. \quad .. \quad .. \quad .. \quad (22)$$

$$\frac{\rho}{p_0}\frac{\partial p_0}{\partial \rho} = \frac{1}{\theta_0}\frac{\partial \theta_0}{\partial \lambda^2}\left[\rho a^3\frac{\partial \lambda^2}{\partial(\rho a^3)}\right], \quad .. \quad .. \quad .. \quad (23)$$

$$\frac{\rho}{\Delta}\frac{\partial \Delta}{\partial \rho} = \frac{\{2\theta_0^3+\lambda^2(\sin\theta_0+\theta_0\cos\theta_0)\}\dfrac{\partial \theta_0}{\partial \lambda^2}+\theta_0\sin\theta_0}{\theta_0^4+2\lambda^2\theta_0\sin\theta_0}\left[\rho a^3\frac{\partial \lambda^2}{\partial(\rho a^3)}\right], \quad .. \quad (24)$$

and

$$\frac{\rho}{\mu}\frac{\partial \mu}{\partial \rho} = \frac{\rho}{\Delta}\frac{\partial \Delta}{\partial \rho} - \frac{\{12\theta_0+\lambda^2(-3\sin\theta_0-\theta_0\cos\theta_0)\}\dfrac{\partial \theta_0}{\partial \lambda^2}+(2\cos\theta_0-\theta_0\sin\theta_0)}{6\theta_0^2+\lambda^2(2\cos\theta_0-\theta_0\sin\theta_0)}\times$$
$$\left[\rho a^3\frac{\partial \lambda^2}{\partial(\rho a^3)}\right]. \quad .. \quad (25)$$

† The density dependence of the parameters p_0^*, Δ^* and μ^* may also be studied in a similar way. This, however, is comparatively less important.

TABLE I

	Theoretical Value			Experimental Value
	$a = 2{\cdot}2\mathring{A}$	$a = 2{\cdot}1\mathring{A}$	$a = 2{\cdot}0\mathring{A}$	
C, (M/s)	322	320	313	238
$p_0/\hbar$, $(\mathring{A}^{-1})$	1·84	1·88	1·92	1·92
Δ/k (°K)	4·6	8·43	11·96	8·70
μ/m	0·04	0·08	0·14	0·16
$p_0^*/\hbar$ $(\mathring{A}^{-1})$	1·01	1·07	1·15	1·11
Δ^*/k (°K)	16·10	16·90	17·62	13·94
μ^*/m	0·14	0·16	0·18	0·53
$\frac{\rho}{c}\frac{\partial c}{\partial \rho}$	0·37	0·44	0·47	2·2
$\frac{\rho}{p_0}\frac{\partial p_0}{\partial \rho}$	0·14	0·19	0·24	0·26
$\frac{\rho}{\Delta}\frac{\partial \Delta}{\partial \rho}$	−6·87	−2·41	−1·16	−0·57
$\frac{\rho}{\mu}\frac{\partial \mu}{\partial \rho}$	−8·0	−3·9	−3·0	−1·8

Here

$$\frac{\partial \theta_0}{\partial \lambda^2} = \frac{\sin\theta_0 + \theta_0 \cos\theta_0}{\lambda^2(\theta_0 \sin\theta_0 - 2\cos\theta_0) - 6\theta_0^2} \quad .. \quad .. \quad .. \quad (26)$$

and $\frac{\partial \lambda^2}{\partial(\rho a^3)}$ may be read from Fig. 1 of reference 3.

The continuous variations of the parameters p_0, Δ and μ with (ρa^3) are given in Figs. 1, 2 and 3.

It is seen from the table that the theoretical results are in qualitative agreement with the experimental ones. A quantitative agreement, however, cannot be expected from such an energy spectrum as it has been obtained under the simplification of the actual interparticle potential by a hard-sphere interaction. The actual numerical evaluation of the macroscopic properties of this gas for the purpose of comparison with the observed values for liquid helium II, therefore, seems to be hardly necessary.

III

Next, for the sake of completeness, we study the thermodynamical properties of the gas in those cases in which the energy spectrum (1) becomes effectively monotonic (cf. Lee, Huang and Yang, 1957). This happens when either (i) the mean momentum of the excitations is so small that the term $\sin\theta/\theta$ is only slightly different from unity and hence its oscillatory character is not of much influence

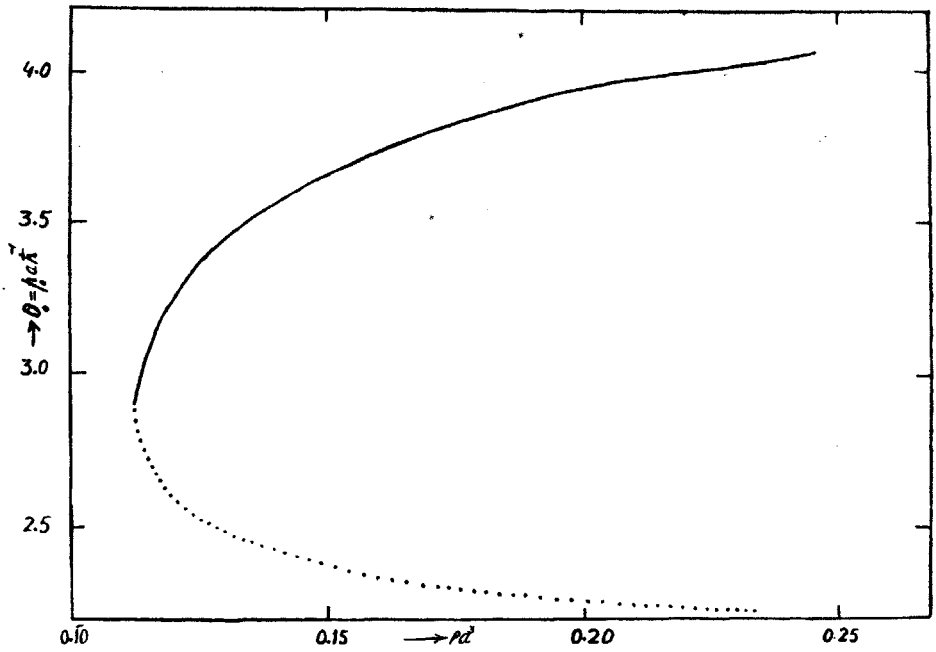
4·0
3·5
3·0
2·5
0·10
0·15
0·20
0·25
ρa^3

FIG. 1

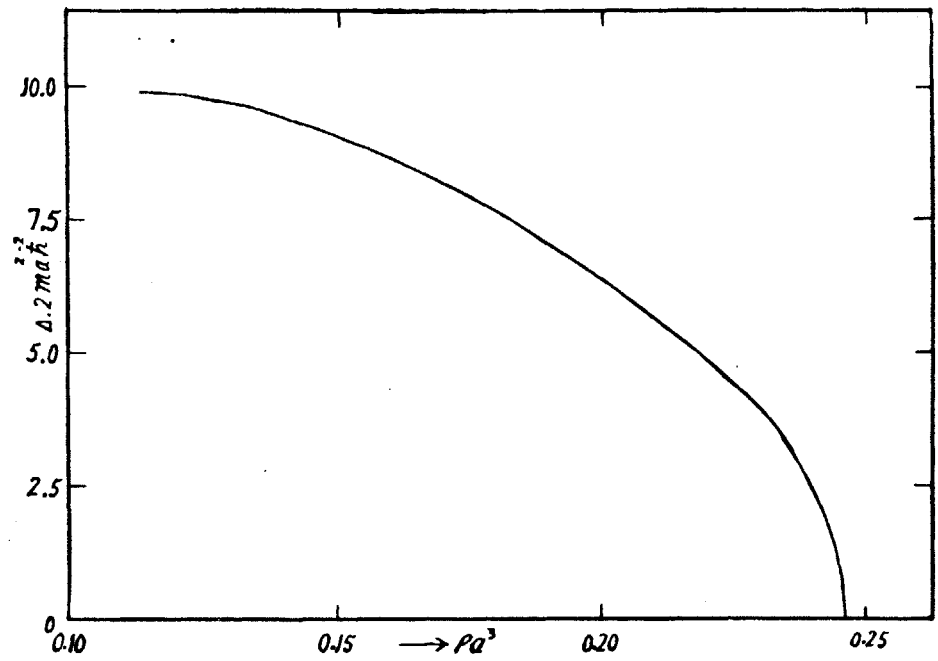
10·0
7·5
5·0
2·5
0
0·10
0·15
0·20
0·25
ρa^3

FIG. 2

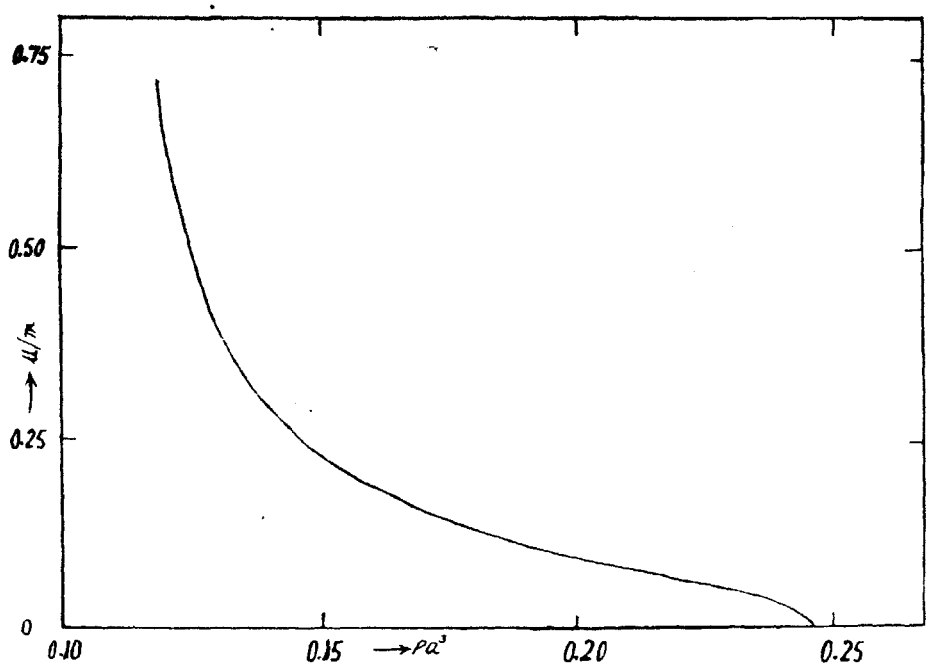

FIG. 3

or (ii) λ^2 is so small that the term p^2 in (1) dominates over the second term. In both these cases equation (2) † may effectively be written as

$$\epsilon = \frac{p}{2m}(p^2+q^2)^{\frac{1}{2}}, \quad .. \quad .. \quad .. \quad .. \quad (27)$$

where

$$q^2 = 2\lambda^2 \frac{\hbar^2}{a^2}\eta. \quad .. \quad .. \quad .. \quad .. \quad (28)$$

Here, η stands for the mean value of the oscillating term $\sin\theta/\theta$ averaged over an assembly of the excitations in question. In the case of an assembly of phonons which corresponds to the case (i) above we have

$$\eta = \sum_{n=0}^{\infty} (-1)^n (n+1) \frac{\zeta(2n+3)}{\zeta(3)} \left(\frac{akT}{c\hbar}\right)^{2n} \quad .. \quad .. \quad .. \quad (29)$$

† For small values of λ^2, i.e. $\rho a^3 \ll 1$, equation (2) yields

$$\lambda^2 = 8\pi\rho a^3. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2a)$$

Further if the momenta are also low ($pa/\hbar \ll 1$), η becomes very nearly equal to unity. The spectrum (27) then reduces precisely to that of Lee, Huang and Yang. It appears worth while to point out here that if one assumes (2a) to be true for higher densities also, as is done by Elenskii and Zyrianov (*Jour. Expt. Theor. Physics*, U.S.S.R., **34**, 530 (1958)), its value in the case of liquid helium II comes out to be about 6, which value cannot give the non-monotonic character and the spectrum would not show features characteristic of the elementary excitations in liquid helium. In this case, therefore, the proper relation to be used is that expressed in equation (2), which gives for λ^2 a value of 30–40.

Since for all practical purposes $\frac{akT}{c\hbar} \ll 1$, we may write equation (29) as

$$\eta = 1 - \frac{1}{\zeta(3)}\left[1 - \left(1 - \frac{akT}{c\hbar}\right)^{-2}\right]. \quad .. \quad .. \quad .. \quad (30)$$

Again, in the case of a degenerate assembly of free particles which corresponds to case (ii), one gets

$$\eta = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \frac{\zeta(n+\frac{3}{2})}{\zeta(\frac{3}{2})} \left(\frac{mkT}{2}\frac{a^2}{\hbar^2}\right)^n, \quad .. \quad .. \quad (31)$$

which for $\left(\frac{mkT}{2}\frac{a^2}{\hbar^2}\right) \ll 1$, may again be approximated as

$$\eta = 1 - \frac{1}{\zeta(\frac{3}{2})}\left[1 - \exp\left\{-\frac{mkT}{2}\frac{a^2}{\hbar^2}\right\}\right] \quad .. \quad .. \quad .. \quad (32)$$

We may now employ the monotonic spectrum (27) in order to derive expressions for the various thermodynamical properties of the (degenerate) assembly of these excitations. The total excitation energy of the assembly is given by

$$U = \frac{4\pi V}{h^3} \int_0^{\infty} \left[\exp\left\{\frac{p}{2mkT}(p^2+q^2)^{\frac{1}{2}}\right\} - 1\right]^{-1} \frac{p}{2m}(p^2+q^2)^{\frac{1}{2}} p^2\, dp, \quad .. \quad (33)$$

whence the substitutions

$$p = q \sinh \frac{t}{2} \quad .. \quad .. \quad .. \quad .. \quad (34)$$

and

$$x = q^2/4mkT \quad .. \quad .. \quad .. \quad .. \quad (35)$$

yield

$$U = \frac{\pi V q^5}{16mh^3} \sum_{n=1}^{\infty} [S_{\frac{5}{2}}(nx) - S_{\frac{3}{2}}(nx) - 2S_{\frac{1}{2}}(nx)]. \quad .. \quad .. \quad (36)$$

Here, the Lommel function $S_\nu(z)$ stands for (Meijer, 1936)

$$S_\nu(z) = \int_0^{\infty} \exp(-z \sinh t) \sinh \nu t\, dt. \quad .. \quad .. \quad (37)$$

Noting that

$$\frac{\partial}{\partial z} S_\nu(z) = \tfrac{1}{2}[C_{\nu-1}(z) - C_{\nu+1}(z)], \quad .. \quad .. \quad .. \quad (38)$$

where the Lommel function $C_\nu(z)$ stands for

$$C_\nu(z) = \int_0^{\infty} \exp(-z \sinh t) \cdot \cosh \nu t\, dt, \quad .. \quad .. \quad (39)$$

equation (14) yields for the specific heat at constant volume (*see* Watson, 1952).

$$C_v = \frac{\pi V q^5}{32mh^3 T} \sum_{n=1}^{\infty} (nx)[C_{\frac{7}{2}}(nx) - C_{\frac{5}{2}}(nx) - 3C_{\frac{3}{2}}(nx) + 3C_{\frac{1}{2}}(nx)]. \quad .. \quad (40)$$

The Helmholtz free energy F may be found by using the relation

$$F = \frac{4\pi VkT}{h^3}\int_0^\infty ln\left[1-\exp\left(-\frac{\epsilon}{kT}\right)\right]\cdot p^2\,dp, \quad \ldots \quad (41)$$

which yields

$$F = \frac{-\pi Vq^5}{24mh^3}\sum_{n=1}^{\infty}\left[S_{\frac{5}{2}}(nx)-3S_{\frac{3}{2}}(nx)+4S_{\frac{1}{2}}(nx)\right]. \quad \ldots \quad (42)$$

From here, the pressure P comes out to be

$$P = \frac{\pi q^5}{24mh^3}\sum_{n=1}^{\infty}\left[S_{\frac{5}{2}}(nx)-3S_{\frac{3}{2}}(nx)+4S_{\frac{1}{2}}(nx)\right]. \quad \ldots \quad (43)$$

The entropy may be calculated by using the relation $S = \frac{-\partial F}{\partial T}$:

$$S = \frac{\pi Vq^5}{48mh^3T}\sum_{n=1}^{\infty}(nx)\left[C_{\frac{7}{2}}(nx)-3C_{\frac{5}{2}}(nx)+3C_{\frac{3}{2}}(nx)-C_{\frac{1}{2}}(nx)\right]. \quad \ldots \quad (44)$$

Again, for the inertial density ρ_n of the excitations we get

$$\rho_n = \frac{\pi mq^3}{6h^3}\sum_{n=1}^{\infty}(nx)\left[C_{\frac{5}{2}}(nx)-3C_{\frac{3}{2}}(nx)+2C_{\frac{1}{2}}(nx)\right]. \quad \ldots \quad (45)$$

After the expressions for the relevant macroscopic properties of the assembly have been set up, one may study them explicitly for the two cases mentioned above. For very low temperatures, i.e. the phonon case, we have from (35), $x \gg 1$. The Lommel functions $S_\nu(z)$ and $C_\nu(z)$ for this case have the following expansions

$$S_\nu(z) = \nu\left[\frac{1}{z^2}-\frac{(2^2-\nu^2)}{z^4}+\frac{(2^2-\nu^2)(4^2-\nu^2)}{z^6}-\ldots\right], \quad \ldots \quad (46)$$

and

$$C_\nu(z) = \left[\frac{1}{z}-\frac{(1^2-\nu^2)}{z^3}+\frac{(1^2-\nu^2)(3^2-\nu^2)}{z^5}-\ldots\right]. \quad \ldots \quad (47)$$

Making these substitutions in the above expressions and carrying out the summations, the various thermodynamical quantities become

$$U = \frac{4\pi^5k^4T^4}{15h^3c^3}V\left[1-\frac{25}{21}\pi^2\left(\frac{kT}{\eta mc^2}\right)^2+\ldots\right], \quad \ldots \quad (36a)$$

$$C_v = \frac{16\pi^5k^4T^3}{15h^3c^3}V\left[1-\frac{25}{14}\pi^2\left(\frac{kT}{\eta mc^2}\right)^2+\ldots\right], \quad \ldots \quad (40a)$$

$$F = \frac{4\pi^5k^4T^4}{45h^3c^3}V\left[1-\frac{5}{7}\pi^2\left(\frac{kT}{\eta mc^2}\right)^2+\ldots\right], \quad \ldots \quad (42a)$$

$$P = \frac{4\pi^5k^4T^4}{45h^3c^3}\left[1-\frac{5}{7}\pi^2\left(\frac{kT}{\eta mc^2}\right)^2+\ldots\right] \quad \ldots \quad (43a)$$

$$S = \frac{16\pi^5 k^4 T^3}{45h^3c^3} V\left[1-\frac{15}{14}\pi^2\left(\frac{kT}{\eta mc^2}\right)+\ldots\right], \quad \ldots \quad \ldots \quad (44a)$$

$$\rho_n = \frac{16\pi^5 k^4 T^4}{45h^3c^5}\left[1-\frac{5}{2}\pi^2\left(\frac{kT}{\eta mc^2}\right)^2+\ldots\right], \quad \ldots \quad \ldots \quad (45a)$$

where the value of η for this case is given by equation (30). It may be noted here that the main terms outside the square brackets are just those appropriate for a phonon assembly in equilibrium at a temperature T. The second terms inside the brackets represent the corrections consequent upon the hard-sphere interaction.

Next, in the case of particle excitation we have $x \ll 1$. The Lommel functions in this case may be expressed as

$$S_\nu(z) = \frac{\pi}{2\sin\pi\nu}[J_{-\nu}+J_\nu]-\frac{1}{\nu}\left[1-\frac{z^2}{(2^2-\nu^2)}+\frac{z^4}{(2^2-\nu^2)(4^2-\nu^2)}-\ldots\right] \quad (48)$$

and

$$C_\nu(z) = \frac{\pi}{2\sin\pi\nu}(J_{-\nu}-J_\nu)+\frac{1}{1^2-\nu^2}\left[z-\frac{z^3}{(3^2-\nu^2)}+\frac{z^5}{(3^2-\nu^2)(5^2-\nu^2)}-\ldots\right] \quad (49)$$

Here $J_\nu(z)$ and $J_{-\nu}(z)$ are the Bessel functions of the first kind. With these substitutions, we obtain to the first approximation, the following results

$$U = V\tfrac{3}{2}kT\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}}\zeta(\tfrac{5}{2})[1-0{\cdot}649x] \quad \ldots \quad \ldots \quad \ldots \quad (36b)$$

$$C_v = V\tfrac{15}{4}k\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}}\zeta(\tfrac{5}{2})[1-0{\cdot}390x], \quad \ldots \quad \ldots \quad \ldots \quad (40b)$$

$$F = -VkT\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}}\zeta(\tfrac{5}{2})[1-1{\cdot}948x] \quad \ldots \quad \ldots \quad \ldots \quad (42b)$$

$$P = kT\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}}\zeta(\tfrac{5}{2})[1-1{\cdot}948x], \quad \ldots \quad \ldots \quad \ldots \quad (43b)$$

$$S = V\tfrac{5}{2}k\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}}\zeta(\tfrac{5}{2})[1-1{\cdot}169x], \quad \ldots \quad \ldots \quad \ldots \quad (44b)$$

and

$$\rho_n = m\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}}\zeta(\tfrac{3}{2}), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (45b)$$

where x occurring in the correction term is given by equations (28), (32) and (35). Again it may be noted that the terms outside the square brackets are those for the particle excitations and the second terms inside the brackets arise due to the hard-sphere interaction.

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