GENERALIZATIONS OF CODAZZI'S EQUATIONS IN A SUBSPACE IMBEDDED IN A FINSLER MANIFOLD

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ABSTRACT

H. Rund has obtained the Gauss and Codazzi's equations for a hypersurface of a Finsler space. A. Elliopoules has extended those results for subspaces of a Finsler manifold. The object of this paper is to further generalize the results of Elliopoules for a subspace F_n imbedded in F_m. We consider $(m-n)$ congruences of curves associated with F_n and obtain the generalizations of Codazzi's equations.

H. Rund (1956) has obtained the Gauss and Codazzi's equations for a hypersurface of a Finsler space. A. Elliopoules (1959) has extended those results for subspaces of a Finsler manifold. We have further generalized those results by considering an $(m-n)$ congruence of curves associated to a subspace F_n of F_m.

The covariant derivative of a vector X^i with respect to x^k is given by

$$X^i_{,k} = \frac{\partial X^i}{\partial x^k} + \overset{*}{\rho}{}^{i}_{hk} X^h \tag{1.1}$$

where $\overset{*}{\rho}{}^{i}_{hk} \left(\equiv \overset{*}{\rho}{}^{i}_{hk}(x, x')\right)$ are the coefficients of connection.
$i, j, k \ldots$ vary from 1 to m; $\alpha, \beta, \gamma \ldots$ vary from 1 to n and $a, b, c, \ldots$ followed by vertical bars vary from $(n+1) \ldots$ to m.

The two systems of unit vectors normal to F_n at a given point P of F_m are given by the solutions $n^i_{a|}$ and $\overset{*}{n}{}^i_{a|}$ of the equations

$$n_{a|i} X^i_\alpha = g_{ij}(x, n_{a|}) n^j_{a|} X^i_\alpha = 0 \tag{1.2}$$

$$\overset{*}{n}_{a|i} X^i_\alpha = g_{ij}(x, x') \overset{*}{n}{}^j_{a|} X^i_\alpha = 0 \tag{1.3}$$

where

$$X^i_\alpha = \frac{\partial x^i}{\partial u^\alpha} \, \{x^i = x^i(u^\alpha)\}$$

and

$$\left| X^i_\alpha \right| \text{ is of rank } n.$$

The two sets are normalized by the following relations

$$g_{ij}(x, n_{a|}) n^i_{a|} n^j_{a|} = 1 \tag{1.4}$$

$$g_{ij}(x, \overset{*}{n}_{a|}) \overset{*}{n}{}^i_{a|} \overset{*}{n}{}^j_{a|} = 1 \tag{1.5}$$

$n^i_{,s}$ are independent of direction element x' whereas n^{*i} are dependent on it.

The fundamental tensor $g^{(u, u')}_{\alpha\beta}$ of the subspace is given by

$$g_{\alpha\beta}(u, u') = g_{ij}(x, x') X^i_\alpha X^j_\beta . \tag{1.6}$$

The $(m-n)$ tensors independent of direction element are defined by

$$\gamma^{(u)}_{a|\alpha\beta} = g_{ij}(x, na|)\, X^i_\alpha X^j_\beta. \tag{1.7}$$

$X^i_{\alpha\beta}$ the generalized covariant derivative of X^i_α is given by

$$X^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} - X^i_\gamma \rho^{*\gamma}_{\alpha\beta} + \rho^{*i}_{hk} X^h_\alpha X^k_\beta \tag{1.8}$$

where $\rho^{*\gamma}_{\alpha\beta}\ (u, u')$ are the connection coefficients in subspace.

There also exist relations

$$X^i_{\alpha\beta} = \sum_a \Omega^*_{a|\alpha\beta}\, n^{*i}_{a|} \tag{1.9}$$

$$\sum \Omega^*_{a|\alpha\beta} \cos\left(n_{a|}, n^*_{b|}\right) = \Omega_{b|\alpha\beta} \tag{1.10}$$

$$n_{a|i} X^i_{\alpha\beta} = \Omega_{a|\alpha\beta} \tag{1.11}$$

$$X^i_{\alpha\beta} = \sum_a A_{a|\alpha\beta}\, n^i_{a|} + w^i_{\alpha\beta} \tag{1.12}$$

where $\Omega_{a|\alpha\beta}$, $\Omega^*_{a|\alpha\beta}$ are second fundamental forms, the former being independent of direction and the latter dependent on it,

and $\qquad w^i_{\alpha\beta}$ satisfies $n_{a|i} w^i_{\alpha\beta} = 0.$

(2). We consider a set of $(m-n)$ congruences of curves one curve of each of which passes through each point of the subspace F_n. Since the vectors in general are not normal to F_n, the contravariant components of a unit vector in the direction of the curve of congruence $\lambda_{a|}$ may be expressed linearly in terms of X^i_α and the set of normals to F_n (Singal and Behari, 1954). Clearly corresponding to two sets $n^i_{a|}$, $n^{*i}_{a|}$ there will be two different expressions, which we shall denote by $\lambda^i_{a|}$ and $\lambda^{*i}_{a|}$ respectively. Thus

2.1. $$\lambda^i_{a|} = t^\alpha_{a|} X^i_\alpha + \sum_b c_{ab|}\, n^i_{b|}$$

2.2. Now $\qquad \cos\left(n_{b|}, \lambda_{a|}\right)$

is defined by

$$\frac{g_{ij}(x, n_{b|})\, n^j_{b|} \lambda^i_{a|}}{\sqrt{g_{ij}(x, n_{a|})\, n^i_{b|} n^j_{b|}}\ \sqrt{g_{ij}(x, \lambda_{a|})\, \lambda^i_{a|} \lambda^j_{a|}}} \tag{2.3}$$

and multiplying (2.1) by $g_{ij}(x, n_{b|})n^j_{b|}$ we notice that first term of R.H.S. is zero and thus

$$\cos\left(n_{b|}, \lambda_{a|}\right) = c_{ab|} + \sum_c c_{ac|}\, a_{b|c} \qquad (b \neq c)$$

where $\qquad a_{bc|} = \cos\left(n_{b|}, n_{c|}\right)$

As $\lambda^i_{a|}$ is a unit vector we have $g_{ij}(x, \lambda_{a|})\lambda^i_{a|}\lambda^j_{a|} = 1.$

Multiplying (2.1) by $g_{ij}(x, \lambda_{a|})\lambda^j_{a|}$

we have

$$1 = g_{ij}(x, \lambda_{a|}) \left(t^{\alpha}_{a|} X^i_{\alpha} + \sum_b c_{ab|} n^i_{b|}\right) \left(t^{\beta}_{a|} X^j_{\beta} + \sum_c c_{ac|} n^j_{c|}\right) \tag{2.4}$$

$$= t^{\alpha}_{a|} t^{\beta}_{a|} X^i_{\alpha} X^j_{\beta} g_{ij}(x, \lambda_{a|}) + g_{ij}(x, \lambda_{a|}) \sum_b c_{ab|} n^i_{b|} \sum_c c_{ac|} n^j_{c|} \tag{2.5}$$

The other two terms vanish on account of (1.2)

Taking the generalized covariant derivative of (2.1) with respect to u^{β} we have

$$\lambda^i_{a|,\beta} = t^{\alpha}_{a|,\beta} X^i_{\alpha} + t^{\alpha}_{a|} X^i_{\alpha\beta} + \sum_b c_{ab|,\beta} n^i_{b|} + \sum_b c_{ab|} n^i_{b|,\beta} \tag{2.6}$$

Now

$$n^j_{b|,\beta} = -\gamma^{\alpha\delta}_{b|} \Omega_{b|\alpha\beta} X^j_{\delta} - \gamma^{\alpha\delta}_{b|} E_{b|ihk} X^j_{\delta} X^k_{\beta} X^i_{\alpha} n^h_{b|} + \sum_k V^{k|}_{b|\beta} n^j_{k|} \tag{2.7}$$

where

$$E_{b|ijk} = g_{ij,k}(x, n_{b|}) \tag{2.8}$$

and $V^{k|}_{b|\beta}$ are defined by

$$n^j_{b|,\beta} n_{c|j} = \sum_k V^{k|}_{b|\beta} n^j_{k|} n_{c|j} = \sum_k V^{k|}_{b|} a_{kc|} = \sum_k V^{k|}_{b|} \cos(n_{k|}, n_{c|}) \tag{2.9}$$

If $A = |a_{kc}| \neq 0$ and A_{kc} stand for the cofactors of a_{kc} then

$$V^{k|}_{b|\beta} = \sum_c \frac{A_{kc}}{A} n^j_{b|,\beta} n_{c|j} \tag{2.10}$$

Therefore

$$\begin{aligned} \lambda^i_{a|,\beta} &= t^{\alpha}_{a|,\beta} X^i_{\alpha} + t^{\alpha}_{a|} \left(\sum_b n^i_{b|} A_{b|\alpha\beta} + w^i_{\alpha\beta}\right) + \sum_b C_{ab|,\beta} n^i_{b|} \\ &\quad + \sum_b C_{ab|} \left[-\gamma^{\alpha\delta}_{b|} \Omega_{b|\alpha\delta} X^i_{\delta} - E_{b|jhk} X^i_{\delta} X^k_{\beta} X^j_{\alpha} n^h_{b|} + \sum_k V^{k|}_{b|\beta} n^i_{k|}\right] \\ &= \left(t^{\delta}_{a|\beta} - \sum_b C_{ab|} \gamma^{\alpha\delta}_{b|} - E_{b|jhk} \gamma^{\alpha\delta}_{b|} X^k_{\beta} X^j_{\alpha} n^h_{b|}\right) X^i_{\delta} \\ &\quad + \sum_b \left(t^{\alpha}_{a|} A_{b|\alpha\beta} + C_{ab|,\beta} + \sum_d C_{ad|} V^{b|}_{d|\beta}\right) n^i_{b|} + t^{\alpha}_{a|} w^i_{\alpha\beta} \\ &= q^{\delta}_{a|\beta} X^i_{\delta} + \sum_b v_{ab|\beta} n^i_{b|} + t^{\alpha}_{a|} w^i_{\alpha\beta} \end{aligned} \tag{2.11}$$

where we have put

$$q^{\delta}_{a|\beta} = t^{\delta}_{a|\beta} - \sum_b C_{ab|} \gamma^{\alpha\delta}_{b|} \Omega_{b|\alpha\beta} - E_{b|jhk} \gamma^{\alpha\delta}_{b|} X^k_{\beta} X^j_{\alpha} n^h_{b|} \tag{2.12}$$

and

$$v_{ab|\beta} = t^{\alpha}_{a|} \Omega_{b|\alpha\beta} + C_{ab|,\beta} + \sum_d C_{ad|} V^{b|}_{d|\beta} \tag{2.13}$$

where $v_{ab|\beta}$ are given by the following equations

$$g_{ij}(x, n_{d|})n^j_{d|}\lambda^i_{a|,\beta} = \sum_d v_{ad|\beta} n^j_{b|} g_{ij}(x, n_{d|}) n^j_{d|}$$

Taking the generalized covariant derivative of (2.11) with respect to u^γ we get

$$\lambda^i_{a|,\beta\gamma} = q^\delta_{a|\beta,\gamma} X^i_\delta + q^\delta_{a|\beta} X^i_{\delta\gamma} + \sum_b v_{ab|\beta,\gamma} n^i_{b|} + \sum_b v_{ab|\beta} n^i_{b|,\gamma} + t^\alpha_{a|,\gamma} w^i_{\alpha\beta} + t^\alpha_{a|} w^i_{\alpha\beta\gamma} \tag{2.14}$$

Applying (2.7) we have

$$\lambda^i_{a|,\beta\gamma} = \left(q^\delta_{a|\beta,\gamma} - \sum_b V_{ab|}\gamma^{\alpha\delta}_{b|}\Omega_{b|\alpha\gamma} - \sum_b V_{ab|\beta}\gamma^{\alpha\delta}_{b|} E_{b|jhk} X^k_\gamma X^j_\alpha n^h_{b|}\right) X^i_\delta + \sum_b \left(q^\delta_{a|\beta} A_{b|\delta\gamma} + V_{ab|\beta,\gamma} + \sum_c V_{ac|\beta} V^{b|}_{c|\beta}\right) n^i_{b|} \tag{2.15}$$

Now

$$\lambda^i_{a|,\beta\gamma} - \lambda^i_{a|,\gamma\beta} = R^i_{jkl}\lambda^j_{a|} X^k_\beta X^l_\gamma \tag{2.16}$$

where R^i_{jkl} is the curvature tensor in F_m defined by

$$\frac{\partial \rho^{*i}_{hl}}{\partial x^k} + \frac{\partial \rho^{*i}_{hl}}{\partial x'^j}\frac{\partial x'^j}{\partial x^k} - \frac{\partial \rho^{*i}_{hk}}{\partial x^l} - \frac{\partial \rho^{*i}_{hk}}{\partial x'^j}\frac{\partial x'^j}{\partial x^l} + \rho^{*i}_{lj}\rho^{*j}_{hk} - \rho^{*i}_{kj}\rho^{*j}_{hl}$$

With the help of (2.15), we have

$$\begin{aligned} R^i_{jkl}\lambda^j_{a|} X^k_\beta X^l_\gamma = {} & (q^\delta_{a|\beta,\gamma} - q^\delta_{a|\gamma,\beta}) X^i_\delta \\ & + \sum_b \gamma^{\alpha\delta}_{b|}(v_{ab|\gamma}\Omega_{b|\alpha\beta} - v_{ab|\beta}\Omega_{b|\alpha\gamma} + v_{ab|\gamma} E_{b|jhk} X^k_\beta X^j_\alpha n^h_{b|} \\ & \qquad - v_{ab|\beta} E_{b|jhk} X^k_\gamma X^j_\alpha n^h_{b|}) X^i_\delta \\ & + \sum_b \Big(q^\delta_{a|\beta} A_{b|\delta\gamma} - q^\delta_{a|\gamma} A_{b|\delta\beta} + v_{ab|\beta,\gamma} - v_{ab|\gamma,\beta} \\ & \qquad + \sum_c v_{ac|\beta} V^{b|}_{c|\beta} - \sum_c v_{ac|\gamma} V^{b|}_{c|\gamma}\Big) n^i_{b|} \\ & \qquad + (q^\delta_{a|\beta} w^i_{\delta\gamma} - q^\delta_{a|\gamma} w^i_{\delta\beta}) \\ & \qquad + (t^\alpha_{a|,\gamma} - t^\alpha_{a|,\beta}) w^i_{\alpha\beta} \\ & \qquad + t^\alpha_{a|}(w^i_{\alpha\beta\gamma} - w^i_{\alpha\gamma\beta}) \end{aligned} \tag{2.17}$$

where $A_{a|\alpha\beta}$ are given by the equations

$$\Omega_{a|\alpha\beta} = n_{a|i} X^i_{\alpha\beta} \sum_b A_{b|\alpha\beta} \cos(n_{a|}, n_{b|}). \tag{2.18}$$

The equations (2.17) are the generalization of Codazzi equations.

Multiplying (2.17) by $g_{hi}(x, n_{d|}) n^h_{d|}$

and applying (1.2) and (1.4) we obtain:

(2.19)
$$g_{hi}(x, n_{d|}) R^i_{jkl} n^h_{d|} \lambda^j_{a|} X^k_\beta X^l_\gamma = \sum_d \Big(q^\delta_{a|\beta} A_{d|\delta\gamma} - q^\delta_{a|\gamma} A_{d|\delta\beta}$$
$$+ v_{ad|\beta,\gamma} - v_{ad|\gamma,\beta} + \sum_c v_{ac|\beta} V^{d|}_{c|\beta} - \sum_c v_{ac|\gamma} V^{d|}_{c|\gamma} \Big) \cos(n_{b|}, n_{d|})$$
$$+ t^\alpha_{a|} (w^i_{\alpha\beta\gamma} - w^i_{\alpha\gamma\beta}) g_{ij}(x, n_{d|}) n^k_{d|}$$

which are alternative forms of Codazzi equations.

N.B.—The last term involving $w^i_{\alpha\beta\gamma}$ can be kept in terms of

$$A_{\alpha\beta} \text{ or } \Omega_{\alpha\beta}$$

3. If $\lambda^i_{a|}$ has no components along various n^i then

(3.1)
$$\lambda^i_{a|} = t^\alpha_{a|} X^i_\alpha$$

i.e. corresponding to congruence denoted by $(a|)$ it is a vector whose components are $\lambda^i_{a|}$ in F_m and $t^\alpha_{a|}$ in F_n.

In view of (3.1) equations (2.15) are of the form

(3.2)
$$\lambda^i_{a|,\beta\gamma} = t^\delta_{a|,\beta\gamma} X^i_\delta + t^\delta_{a|,\beta} X^i_{\delta\gamma}$$
$$+ \sum_b t^\alpha_{a|} A_{b|\alpha\beta,\gamma} n^i_{b|} + \sum_b t^\alpha_{a|,\gamma} A_{b|\alpha\beta} n^i_{b|}$$
$$+ \sum_b t^\alpha_{a|} A_{b|\alpha\beta} n^i_{b|,\gamma} + \sum_b t^\alpha_{a|,\gamma} A_{b|\alpha\beta} n^i_{b|}$$
$$+ \sum_b t^\alpha_{a|} A_{b|\alpha\beta} n^i_{b|,\gamma} + t^\alpha_{a|,\gamma} w^i_{\alpha\beta} + t^\alpha_{a|} w^i_{\alpha\beta\gamma}$$

or

(3.3)
$$\lambda^i_{a|,\beta\gamma} - \lambda^i_{a|,\gamma\beta} = (t^\delta_{a|,\beta\gamma} - t^\delta_{a|,\gamma\beta}) X^i_\delta$$
$$+ (t^\alpha_{a|,\beta} X^i_{\alpha\gamma} - t^\alpha_{a|,\gamma} X^i_{\alpha\beta} + t^\alpha_{a|,\gamma} w^i_{\alpha\beta} - t^\alpha_{a|,\beta} w^i_{\alpha\gamma})$$
$$+ t^\alpha_{a|} \sum_b (A_{b|\alpha\beta,\gamma} n^i_{b|} - A_{b|\alpha\gamma,\beta} n^i_{b|})$$
$$+ (t^\alpha_{a|,\gamma} \sum_b A_{b|\alpha\beta} n^i_{b|} - t^\alpha_{a|,\beta} \sum_b A_{b|\alpha\gamma} n^i_{b|})$$
$$+ t^\alpha_{a|} \Big(\sum_b A_{b|\alpha\beta} n^i_{b|,\gamma} - \sum_b A_{b|\alpha\gamma} n^i_{b|,\beta} \Big)$$
$$+ t^\alpha_{a|} (w^i_{\alpha\beta\gamma} - w^i_{\alpha\gamma\beta}).$$

Since

$$w^i_{\alpha\beta} = X^i_{\alpha\beta} - \sum_a A_{a|\alpha\beta} n^i_{a|}$$

the second term on R.H.S. of (3.3) can be simplified and we have

$$R^i_{jkl}\lambda^j_{a|}X^k_\beta X^l_\gamma = R^\delta_{\alpha\beta\gamma}t^\alpha_{a|}X^i_\delta + t^\alpha_{a|,\beta}\sum_b A_{b|\alpha\gamma}n^i_b + t^\alpha_{a|,\gamma}\sum_b A_{b|\alpha\beta}n^i_{b|} - t^\alpha_{a|,\beta}\sum_b A_{b|\alpha\gamma}n^i_{b|} + t^\alpha_{a|}\left(\sum_b A_{b|\alpha\beta}n^i_{b|,\gamma} - \sum_b A_{b|\alpha\gamma}n^i_{b|,\beta}\right) + t^\alpha_{a|}(w^i_{\alpha\beta\gamma} - w^i_{\alpha\gamma\beta}). \tag{3.4}$$

If we further assume that vectors $\lambda^i_{a|}$ and $t^\alpha_{a|}$ are tangential, we have

$$x'^i = u'^\alpha X^i_\alpha \quad .. \quad .. \quad .. \quad .. \quad \text{(from 3.1)}$$

Accordingly (3.4) reduces to

$$R^i_{jkl}x'^j X^k_\beta X^l_\gamma = R^\delta_{\alpha\beta\gamma}u'^\alpha X^i_\delta + u'^\alpha(A_{b|\alpha\beta,\gamma}n^i_{b|} - A_{b|\alpha\gamma,\beta}n^i_{b|}) + \left(\sum_b A_{b|\alpha\beta}n^i_{b|,\gamma} - \sum_b A_{b|\alpha\gamma}n^i_{b|,\beta}\right)u'^\alpha + (w^i_{\alpha\beta\gamma} - w^i_{\alpha\gamma\beta})u'^\alpha$$

or

$$R^i_{jkl}X^j_\alpha X^k_\beta X^l_\gamma = R^\delta_{\alpha\beta\gamma}X^i_\delta + (A_{b|\alpha\beta,\gamma}n^i_{b|} - A_{b|\alpha\gamma,\beta}n^i_{b|}) + \sum_b\left(A_{b|\alpha\beta}n^i_{b|,\gamma} - A_{b|\alpha\gamma}n^i_{b|,\beta}\right) + \left(w^i_{\alpha\beta\gamma} - w^i_{\alpha\gamma\beta}\right) \tag{3.5}$$

The equation (3.5) is the same as given by Elliopoules in (3) as Codazzi equations.

4. Every $n^i_{a|}$ can be decomposed as having components along n^{*i} and X^i_α, therefore the unit vector parallel to the above-mentioned congruence can be decomposed in terms of X^i_α and $n^{*i}_{a|}$. We denote this decomposition by $\lambda^{*i}_{a|}$.

Let

$$\lambda^{*i}_{a|} = t^{*\alpha}_{a|}X^i_\alpha + \sum_b C^*_{ab|}n^{*i}_{b|} \tag{4.1}$$

where

$$C^*_{ab|} = \cos(n^*_{b|}, \lambda^*_{a|}) \tag{4.2}$$

is given by

$$g_{ij}(x, x')n^{*j}_{b|}\lambda^{*i}_{a|} = C^*_{ab|} \tag{4.3}$$

on account of (1.3) the other terms are zero.

As

$$g_{ij}(x, \lambda^{*i}_{a|})\lambda^{*i}_{a|}\lambda^{*j}_{a|} = 1$$

$t^{*\alpha}_{a|}$ is given by the following relation

$$1 = t^{*\alpha}_{a|} t^{*\beta}_{a|} X^i_\alpha X^j_\beta g_{ij}(x, \lambda^*_{a|}) + g_{ij}(x, \lambda^*_{a|}) \sum_b C^*_{ab|} n^{*i}_{b|} \sum_c C^*_{ac|} n^{*j}_{c|} \tag{4.4}$$

Taking the generalized covariant derivative with respect to u^β we get

$$\lambda^i_{a|,\beta} = t^{*\alpha}_{a|,\beta} X^i_\alpha + t^{*\alpha}_{a|} X^i_{\alpha\beta} + \sum_b C^*_{ab|,\beta} n^{*i}_{b|} + \sum_b C^*_{ab|} \left(B^\delta_{b|\beta} X^i_\delta + \sum_k N^{b|}_{k|\beta} n^{*i}_{k|} \right) \tag{4.5}$$

which we can put as

$$\lambda^{*i}_{a|,\beta} = t^{*\alpha}_{a|,\beta} X^i_\alpha + t^{\alpha}_{a|} \sum_b \Omega^*_{b|\alpha\beta} n^{*i}_{b|} + \sum_b C^*_{ab|,\beta} n^{*i}_{b|} + \sum_b C^*_{ab|} \left(B^\delta_{b|\beta} X^i_\delta + \sum_k N^{b|}_{k|\beta} n^{*i}_{k|} \right)$$

where

$$B^\epsilon_{a|\beta} = -\psi_{a|} \Omega^*_{a|\alpha\beta} g^{\alpha\epsilon} - E^{*(xx')}_{ihk} g^{\alpha\epsilon} X^k_\beta X^i_\alpha n^{*h}_{a|} \tag{4.6}$$

where

$$E^*_{ihk} = g^{(x, x')}_{ih, k}$$

and the $N_{,s}$ satisfy the following relations

$$n^{*j}_{a|,\beta} n^*_{b|j} = N^a_{b|\beta} \psi_{a|} \tag{4.7}$$

$\psi_{a|}$ being given by

$$\psi_{a|} \delta^{a|}_{b|} = g_{ij}(x, x') n^{*i}_{a|} n^{*j}_{b|}.$$

Separating the terms containing X^i_α and $n^{*i}_{a|}$ we have

$$\lambda^{*i}_{a|,\beta} = \left(t^{*\delta}_{a|\beta} + \sum_b C^*_{ab|} B^\delta_{b|\beta} \right) X^i_\delta + \sum_b \left(t^{*\alpha}_{a|} \Omega^*_{b|\alpha\beta} + C^*_{ab|,\beta} + \sum_k C^*_{ak|} N^{k|}_{b|\beta} \right) n^{*i}_{b|} \tag{4.8}$$

or

$$\lambda^{*i}_{a|,\beta} = q^{*\delta}_{a|\beta} X^i_\delta + \sum_b v^*_{ab|\beta} n^{*i}_{b|} \tag{4.9}$$

where we have put

$$q^{*\delta}_{a|\beta} = t^{*\delta}_{a|,\beta} + \sum_b C^*_{ab|} B^\delta_{b|\beta} \tag{4.10}$$

and

$$v^*_{ab|\beta} = t^{*\alpha}_{a|} \Omega^*_{b|\alpha\beta} + C^*_{ab|,\beta} + \sum_k C^*_{ak|} N^{k|}_{b|\beta} \tag{4.10a}$$

We take the generalized covariant derivative of (4.9) and since

$$\lambda^{*i}_{a|,\beta\gamma} - \lambda^{*i}_{a|,\gamma\beta} = R^{i}_{jkl}\lambda^{*j}_{a|}X^{k}_{\beta}X^{l}_{\gamma}$$

we have

$$R^{i}_{jkl}\lambda^{*j}_{a|}X^{k}_{\beta}X^{l}_{\gamma} = (q^{*\delta}_{a|\beta,\gamma} - q^{*\delta}_{a|\gamma,\beta})X^{i}_{\delta} + \Big(\sum_{b} v^{*}_{ab|\beta}B^{\delta}_{b\,\gamma} - \sum_{b} v^{*}_{ab|\gamma}B^{\delta}_{b|\beta}\Big)X^{i}_{\delta} \tag{4.11}$$
$$+ \sum_{b}\Big[(v^{*}_{ab|\beta,\gamma} - v^{*}_{ab|\gamma,\beta}) + \Big(\sum_{k} v^{*}_{ak|\beta}N^{k|}_{b|\gamma} - \sum_{k} v^{*}_{ak|\gamma}N^{k|}_{b|\beta}\Big)$$
$$+ (q^{*\delta}_{a|\beta}\Omega^{*}_{b|\delta\gamma} - q^{*\delta}_{a|\gamma}\Omega^{*}_{b|\delta\beta})\Big] n^{*i}_{b|}$$

Equations (4.11) are further generalizations of Codazzi's equations. Assuming $\lambda^{*i}_{a|}$ has no components along the normal

we have
$$\lambda^{*i}_{a|} = t^{*\alpha}_{a|}X^{i}_{\alpha}$$

which reduces (4.10) and (4.10*a*) to the following form:

$$q^{*\delta}_{a|\beta} = t^{*\delta}_{a|,\beta}, \quad v^{*}_{ab|\beta} = t^{*\alpha}_{a|}\Omega^{*}_{b|\alpha\beta}$$

Consequently

$$\lambda^{*i}_{a|,\beta\gamma} = t^{*\alpha}_{a|,\beta\gamma}X^{i}_{\alpha} + t^{*\alpha}_{a|,\beta}\sum_{c}\Omega^{*}_{c|\alpha\gamma}n^{*i}_{c|} + t^{*\alpha}_{a|,\gamma}\sum_{b}\Omega^{*}_{b|\alpha\beta}n^{*i}_{b|} \tag{4.12}$$
$$+ t^{*\alpha}_{a|}\sum_{b}\big\{\Omega^{*}_{b|\alpha\beta,\gamma}n^{*i}_{b|} + \Omega^{*}_{b|\alpha\beta}n^{*i}_{b|,\gamma}\big\}$$

Thus equations (4.11) are of the form

$$R^{i}_{jkl}\lambda^{*j}_{a|}X^{k}_{\beta}X^{l}_{\gamma} = R^{\delta}_{\alpha\beta\gamma}t^{*\alpha}_{a|}X^{i}_{\delta} \tag{4.13}$$
$$+ t^{*\alpha}_{a|}\Big(\sum_{b}\Omega^{*}_{b|\alpha\beta}n^{*i}_{b|,\gamma} - \sum_{b}\Omega^{*}_{b|\alpha\gamma}n^{*i}_{b|,\beta}\Big)$$
$$+ t^{*\alpha}_{a|}\Big(\sum_{b}\Omega^{*}_{b|\alpha\beta,\gamma}n^{*i}_{b|} - \sum_{b}\Omega^{*}_{b|\alpha\gamma,\beta}n^{*i}_{b|}\Big)$$

If we further assume that $t^{*\alpha}_{a|}$ and $\lambda^{*i}_{a|}$ are tangential and are thus connected by the relation

$$x'^{i} = u'^{\alpha}X^{i}_{\alpha} \tag{4.14}$$

(4.13) takes the form

$$R^{i}_{jkl}X^{j}_{\alpha}X^{k}_{\beta}X^{l}_{\gamma} = R^{\delta}_{\alpha\beta\gamma}X^{i}_{\delta} \tag{4.15}$$
$$+ \Big(\sum_{b}\Omega^{*}_{b|\alpha\beta}n^{*i}_{b|,\gamma} - \sum_{b}\Omega^{*}_{b|\alpha\gamma}n^{*i}_{b|,\beta}\Big)$$
$$+ \Big(\sum_{b}\Omega^{*}_{b|\alpha\beta,\gamma}n^{*i}_{b|} - \sum_{b}\Omega^{*}_{b|\alpha\gamma,\beta}n^{*i}_{b|}\Big)$$

which are the same as Elliopoules' equations connecting the curvature tensor of subspace F_n with that of F_m. When $m = n+1$ the equations (3.5) and (4.15) reduce to Codazzi equations for hypersurfaces as given by H. Rund (1956).

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