ON THE ABSOLUTE HARMONIC SUMMABILITY OF A DOUBLE SERIES ASSOCIATED WITH THE DOUBLE FOURIER SERIES

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ABSTRACT

In this paper we have investigated the absolute harmonic summability of a double series associated with the double Fourier series. The theorem proved extends a result of Varshney (1959) to double series.

1. Definition 1.—A double series $\Sigma\Sigma U_{m,n}$ with the sequence of partial sums $\{s_{m,n}\}$ is said to be summable by harmonic means or summable $(H, 1, 1)$ if the sequence

$$\sigma_{m,n} = \frac{1}{\rho_m \rho_n} \sum_{l=0}^{m} \sum_{k=0}^{n} \frac{s_{m-l,n-k}}{(l+1)(k+1)}, \tag{1.1}$$

$$\left(\rho_n = 1 + \tfrac{1}{2} + \ldots + \frac{1}{n+1}\right),$$

tends to a finite limit as $(m, n) \to \infty$.

This is known (Sharma 1958). It may be noted that this is a particular case $p_n = \frac{1}{n+1}$ of Nörlund summability of the double sequence as defined by Herriot (1942).

Definition 2.—A double series $\Sigma\Sigma U_{m,n}$ is said to be absolutely summable by harmonic means or summable $|H, 1, 1|$ if

$$\sum_m \sum_n |\sigma_{m,n} - \sigma_{m,n-1} - \sigma_{m-1,n} + \sigma_{m-1,n-1}| < \infty,$$

$$\sum_n |\sigma_{m,n} - \sigma_{m,n-1}| < \infty,$$

$$\sum_m |\sigma_{m,n} - \sigma_{m-1,n}| < \infty.$$

Compare Timan (1956).

2. Suppose that the function $f(u, v)$ is integrable in the sense of Lebesgue over the square $(-\pi, \pi; -\pi, \pi)$ and is periodic with period 2π in each variable. The double Fourier series associated with the function $f(u, v)$ is

$$\Sigma\Sigma A_{m,n}(u, v). \tag{2.1}$$

We write

$$\phi(u, v) = \tfrac{1}{4}[f(x+u, y+v)+f(x+u, y-v)+f(x-u, y+v)+f(x-u, y-v)],$$

$$h(n, u) = \frac{1}{\rho_n \rho_{n-1}} \sum_{k=0}^{n-1}{}' \left(\frac{\rho_n}{k+1} - \frac{\rho_k}{n+1}\right) \frac{\cos(n-k)u}{\log(n-k+1)},$$

$$g_1(n, u) = \frac{\cos nu}{\rho_n \log(n+1)},$$

$$g_2(n, u) = \frac{1}{(n+1)\rho_n \rho_{n-1}} \sum_{k=1}^{n-1}{}' \frac{\cos(n-k)u}{(k+1)\log(n-k+1)},$$

$$g_3(n, u) = \frac{1}{\rho_{n-1}} \sum_{k=1}^{n-1}{}' \frac{\cos(n-k)u}{k(k+1)\log(n-k+1)}.$$

We prove the following theorem:

THEOREM. *If each of*

$$\frac{\partial^2 \phi(u, v)}{\partial u \partial v}, \quad \frac{\partial \phi(u, v)}{\partial u}, \quad \frac{\partial \phi(u, v)}{\partial v},$$

is finite and of a constant sign in the rectangle $(0, \pi; 0, \pi)$ *then the double series*

$$\sum_1^\infty \sum_1^\infty \frac{A_{m,n}(u, v)}{\log(m+1) \cdot \log(n+1)}$$

is summable $|H, 1, 1|$.

This extends a result of Varshney (1959) to double series.

3. We shall require the following lemmas:

LEMMA 1. *Uniformly for* $0 < t < \pi$

$$\left|\sum_m^n \frac{\sin kt}{k}\right| \leqslant A$$

where m and n are any positive integers.

This is known (Titchmarsh 1939, p. 440).

LEMMA 2. *If* $0 < t < \pi$, *then*

$$\left|\sum_{k=0}^m \frac{\cos(k+1)t}{k+1}\right| = O\left(1+\log\frac{1}{t}\right).$$

This is known (Hardy and Rogosinski 1947).

LEMMA 3. *If* $0 < t < \pi$, *then for all positive integers a and b,*

$$\sum_{k=a}^b \frac{\sin(n-k)t}{k+1} = O\left(1+\log\frac{1}{t}\right).$$

This follows from lemmas 1 and 2.

LEMMA 4. *If*

$$\rho_n = 1+\tfrac{1}{2}+\ldots+\frac{1}{n+1}$$

then uniformly for $0 < t < \pi$ *we have*

$$\sum_n \left| \frac{1}{\rho_n \rho_{n-1}} \sum_{k=0}^{n-1} \left(\frac{\rho_n}{k+1} - \frac{\rho_k}{n+1} \right) \frac{\sin (n-k)t}{(n-k) \log (n-k+1)} \right| < \infty.$$

This is known (Varshney 1959).

LEMMA 5. *We have*

(i) $$\sum_{k=1}^{[n/2]-2} \left| \Delta \left(\frac{1}{(n-k) \log (n-k+1)} \right) \right| = O(n^{-1}).$$

(ii) $$\sum_{k=1}^{[n/2]-2} \left| \Delta \left(\frac{1}{k \log (n-k+1)} \right) \right| = O\left(\frac{1}{\log n} \right).$$

Proof: For proving (i) we observe that

$$\sum_{k=1}^{[n/2]-2} \left| \Delta \left(\frac{1}{(n-k) \log (n-k+1)} \right) \right|$$

$$\leqslant \sum_{k=1}^{[n/2]-2} \frac{1}{(n-k)^2 \log (n-k+1)} + \sum_{k=1}^{[n/2]-2} \frac{1}{(n-k)(n-k+1) \log^2 (n-k+1)}$$

$$= O\left[\sum_{k=1}^{[n/2]-2} \frac{1}{(n-k)^2 \log (n-k+1)} \right] = O\left[\sum_{k=1}^{[n/2]-2} \frac{1}{(n-k)^2} \right]$$

$$= O\left(\frac{1}{n} \right).$$

Again

$$\sum_{k=1}^{[n/2]-2} \left| \Delta \left(\frac{1}{k \log (n-k+1)} \right) \right|$$

$$\leqslant \sum_{k=1}^{[n/2]-2} \frac{1}{k^2 \log (n-k+1)} + \sum_{k=1}^{[n/2]-2} \frac{1}{k(n-k+1) \log^2 (n-k+1)}$$

$$= O\left[\frac{1}{\log n} \sum_{k=1}^{[n/2]-2} \frac{1}{k^2} \right] + O\left[\sum_{k=1}^{[n/2]-2} \frac{1}{n \log^2 n} \right]$$

$$= O\left[\frac{1}{\log n} \right].$$

This proves the lemma.

4. Proof of the theorem. Since

$$\sigma_{m,\,n}=\frac{1}{\rho_m\rho_n}\sum_{\mu=0}^{m}\sum_{k=0}^{n}\frac{s_{m-\mu,\,n-k}}{(\mu+1)\,(k+1)}$$

$$=\frac{1}{\rho_m\rho_n}\left[\sum_{k=0}^{n}\frac{(m+1)^{-1}s_{0,\,n-k}+m^{-1}s_{1,\,n-k}+\ldots+s_{m,\,n-k}}{k+1}\right]$$

$$=\frac{1}{\rho_m\rho_n}\left[(m+1)^{-1}\sum_{k=0}^{n}\frac{s_{0,\,n-k}}{k+1}+m^{-1}\sum_{k=0}^{n}\frac{s_{1,\,n-k}}{k+1}+\ldots+\sum_{k=0}^{n}\frac{s_{m,\,n-k}}{k+1}\right]$$

$$=\frac{1}{\rho_m\rho_n}\Big[(m+1)^{-1}\{(n+1)^{-1}s_{0,\,0}+n^{-1}s_{0,\,1}+\ldots+1.s_{0,\,n}\}$$
$$+m^{-1}\{(n+1)^{-1}s_{1,\,0}+n^{-1}s_{1,\,1}+\ldots+1.s_{1,\,n}\}$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+1.\{(n+1)^{-1}s_{m,\,0}+n^{-1}s_{m,\,1}+\ldots+1.s_{m,\,n}\}\Big]$$

$$=\frac{1}{\rho_m\rho_n}\Big[(m+1)^{-1}\{\rho_nU_{0,\,0}+\rho_{n-1}U_{0,\,1}+\ldots+U_{0,\,n}\}$$
$$+m^{-1}\{\rho_nU_{1,\,0}+\rho_{n-1}U_{1,\,1}+\ldots+U_{1,\,n}\}$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+1.\{\rho_nU_{m,\,0}+\rho_{n-1}U_{m,\,1}+\ldots+U_{m,\,n}\}\Big],$$

$$\left(U_{m,\,n}=\frac{A_{m,\,n}(u,\,v)}{\log\,(n+1)\log\,(m+1)}\right)$$

we have

$$\sigma_{m,\,n}-\sigma_{m,\,n-1}=\frac{1}{\rho_m}\Big[(m+1)^{-1}\left\{\sum_{\nu=0}^{n-1}\left(\frac{\rho_\nu}{\rho_n}-\frac{\rho_{\nu-1}}{\rho_{n-1}}\right)U_{0,\,n-\nu}\right\}$$
$$+m^{-1}\left\{\sum_{\nu=0}^{n-1}\left(\frac{\rho_\nu}{\rho_n}-\frac{\rho_{\nu-1}}{\rho_{n-1}}\right)U_{1,\,n-\nu}\right\}$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+1.\left\{\sum_{\nu=0}^{n-1}\left(\frac{\rho_\nu}{\rho_n}-\frac{\rho_{\nu-1}}{\rho_{n-1}}\right)U_{m,\,n-\nu}\right]$$

$$=\left[\sum_{\mu=0}^{m-1}\frac{1}{\rho_m(\mu+1)}\sum_{\nu=0}^{n-1}\left(\frac{\rho_\nu}{\rho_n}-\frac{\rho_{\nu-1}}{\rho_{n-1}}\right)U_{m-\mu,\,n-\nu}\right]$$

$$=\left[\sum_{\mu=0}^{m-1}\frac{1}{\rho_m(\mu+1)}\frac{1}{\rho_n\rho_{n-1}}\sum_{\nu=0}^{n-1}\left(\frac{\rho_n}{\nu+1}-\frac{\rho_\nu}{n+1}\right)U_{m-\mu,\,n-\nu}\right].$$

So that

$$\sigma_{m,n}-\sigma_{m,n-1}-\sigma_{m-1,n}+\sigma_{m-1,n-1}$$

$$=\frac{1}{\rho_m}\frac{1}{\rho_n\rho_{n-1}}\sum_{\nu=0}^{n-1}\left(\frac{\rho_n}{\nu+1}-\frac{\rho_\nu}{n+1}\right)U_{m,n-\nu}$$

$$+\left[\sum_{\mu=1}^{m-1}\left(\frac{1}{(\mu+1)\rho_m}-\frac{1}{\mu\rho_{m-1}}\right)\cdot\frac{1}{\rho_n\rho_{n-1}}\sum_{\nu=0}^{n-1}\left(\frac{\rho_n}{\nu+1}-\frac{\rho_\nu}{n+1}\right)U_{m-\mu,n-\nu}\right]$$

$$=\frac{1}{\rho_m}\frac{1}{\rho_n\rho_{n-1}}\sum_{\nu=0}^{n-1}\left(\frac{\rho_n}{\nu+1}-\frac{\rho_\nu}{n+1}\right)U_{m,n-\nu}$$

$$-\frac{1}{(m+1)\rho_m\rho_{m-1}}\sum_{\mu=1}^{m-1}\frac{1}{\mu+1}\cdot\frac{1}{\rho_n\rho_{n-1}}\sum_{\nu=0}^{n-1}\left(\frac{\rho_n}{\nu+1}-\frac{\rho_\nu}{n+1}\right)U_{m-\mu,n-\nu}$$

$$-\frac{1}{\rho_{m-1}}\sum_{\mu=1}^{m-1}\frac{1}{\mu(\mu+1)}\cdot\frac{1}{\rho_n\rho_{n-1}}\sum_{\nu=0}^{n-1}\left(\frac{\rho_n}{\nu+1}-\frac{\rho_\nu}{n+1}\right)U_{m-\mu,n-\nu}$$

since

$$\left[\frac{1}{(\mu+1)\rho_m}-\frac{1}{\mu\cdot\rho_{m-1}}\right]=\frac{\rho_{m-1}\mu-(\mu+1)\rho_m}{\mu(\mu+1)\rho_m\rho_{m-1}}=\frac{\mu(\rho_{m-1}-\rho_m)-\rho_m}{\mu(\mu+1)\rho_m\rho_{m-1}}$$

$$=-\frac{1}{(m+1)(\mu+1)\rho_m\rho_{m-1}}-\frac{1}{\mu(\mu+1)\rho_{m-1}}.$$

For the double Fourier series of $f(u, v)$ at $u=k$, $v=l$,

$$A_{m,n}(u,v)=\frac{4}{\pi^2}\int_0^\pi\int_0^\pi\phi(u,v)\cos nu\cos mv\,du\,dv$$

so that

$$\sigma_{m,n}-\sigma_{m,n-1}-\sigma_{m-1,n}+\sigma_{m-1,n-1}$$

$$=\frac{4}{\pi^2}\int_0^\pi\int_0^\pi\phi(u,v)\left(\frac{1}{\rho_n\rho_{n-1}}\sum_{k=0}^{n-1}\left(\frac{\rho_n}{k+1}-\frac{\rho_k}{n+1}\right)\frac{\cos(n-k)u}{\log(n-k+1)}\right)$$

$$\left(\frac{\cos mv}{\rho_m\cdot\log(m+1)}\right)du\,dv$$

$$-\frac{4}{\pi^2}\int_0^\pi\int_0^\pi\phi(u,v)\left(\frac{1}{\rho_n\rho_{n-1}}\sum_{k=0}^{n-1}\left(\frac{\rho_n}{k+1}-\frac{\rho_k}{n+1}\right)\frac{\cos(n-k)u}{\log(n-k+1)}\right)$$

$$\left(\frac{1}{(m+1)\rho_m\rho_{m-1}}\cdot\sum_{l=1}^{m-1}\frac{\cos(m-l)v}{(l+1)\log(m-l+1)}\right)du\,dv$$

$$-\frac{4}{\pi^2}\int_0^\pi\int_0^\pi\phi(u,v)\left(\frac{1}{\rho_n\rho_{n-1}}\sum_{k=0}^{n-1}\left(\frac{\rho_n}{k+1}-\frac{\rho_k}{n+1}\right)\frac{\cos(n-k)u}{\log(n-k+1)}\right)$$

$$\left(\frac{1}{\rho_{m-1}}\cdot\sum_{l=1}^{m-1}\frac{\cos(m-l)v}{l(l+1)\log(m-l+1)}\right)du\,dv$$

$$= \frac{4}{\pi^2}\int_0^\pi\int_0^\pi \phi(u, v)h(n, u)g_1(m, v)du\,dv$$

$$-\frac{4}{\pi^2}\int_0^\pi\int_0^\pi \phi(u, v)h(n, u)g_2(m, v)du\,dv$$

$$-\frac{4}{\pi^2}\int_0^\pi\int_0^\pi \phi(u, v)h(n, u)g_3(m, v)du\,dv.$$

In order to show that

$$\sum_m\sum_n \left|\sigma_{m,n}-\sigma_{m,n-1}-\sigma_{m-1,n}+\sigma_{m-1,n-1}\right| < \infty,$$

we have to establish that

$$\sum_m\sum_n \left|\int_0^\pi\int_0^\pi \phi(u, v)h(n, u)g_1(m, v)du\,dv\right| < \infty,$$

$$\sum_m\sum_n \left|\int_0^\pi\int_0^\pi \phi(u, v)h(n, u)g_2(m, v)du\,dv\right| < \infty,$$

$$\sum_m\sum_n \left|\int_0^\pi\int_0^\pi \phi(u, v)h(n, u)g_3(m, v)du\,dv\right| < \infty.$$

On applying integration by parts for double integral (Gergen 1933; Young 1913), we observe that

$$\int_0^\pi\int_0^\pi \phi(u, v)h(n, u)g_r(m, v)du\,dv$$

$$= \phi(\pi, \pi)\left(\int_0^\pi h(n, u)\,du\right)\left(\int_0^\pi g_r(m, v)\,dv\right)$$

$$-\left(\int_0^\pi h(n, u)\,du\right)\left[\int_0^\pi \left\{\frac{\partial}{\partial v}\phi(\pi, v)\cdot\int_0^v g_r(m, v)\,dv\right\}dv\right]$$

$$-\left(\int_0^\pi g_r(m, v)\,dv\right)\left[\int_0^\pi \left\{\frac{\partial}{\partial u}\phi(u, \pi)\cdot\int_0^u h(n, u)\,du\right\}du\right]$$

$$+\int_0^\pi\int_0^\pi \frac{\partial^2\phi(u, v)}{\partial u\,\partial v}\left(\int_0^u h(n, u)\,du\right)\left(\int_0^v g_r(m, v)\,dv\right)du\,dv$$

and

$$\sum_m\left|\int_0^\pi \left\{\frac{\partial}{\partial v}\phi(\pi, v)\cdot\int_0^v g_r(m, v)\,dv\right\}dv\right|$$

$$\leqslant \int_0^\pi |d\phi(\pi, v)|\cdot\left\{\sum_m\left|\int_0^v g_r(m, v)\,dv\right|\right\},$$

$$\sum_n \left| \int_0^\pi \left\{ \frac{\partial}{\partial u} \phi(u, \pi) \cdot \int_0^u h(n, u)\, du \right\} du \right|$$

$$\leqslant \int_0^\pi |d\phi(u, \pi)| \cdot \left\{ \sum_n \left| \int_0^u h(n, u)\, du \right| \right\},$$

$$\sum_m \sum_n \left| \int_0^\pi \int_0^\pi \frac{\partial^2 \phi(u, v)}{\partial u\, \partial v} \left(\int_0^u h(n, u)\, du \right) \left(\int_0^v g_r(m, v)\, dv \right) du\, dv \right|$$

$$\leqslant \int_0^\pi \int_0^\pi |d^2\phi(u, v)| \cdot \left\{ \sum_m \left| \int_0^v g_r(m, v)\, dv \right| \right\} \left\{ \sum_n \left| \int_0^u h(n, u)\, du \right| \right\}.$$

Since by the hypothesis of the theorem we have

$$\int_0^\pi \int_0^\pi \left| \frac{\partial^2 \phi(u, v)}{\partial u \partial v} \right| du\, dv < \infty,$$

$$\int_0^\pi \left| \frac{\partial \phi(u, \pi)}{\partial u} \right| du < \infty, \quad \int_0^\pi \left| \frac{\partial \phi(\pi, v)}{\partial v} \right| dv < \infty,$$

it suffices for our purpose to show that, uniformly for

$$0 < u < \pi, \ 0 < v < \pi,$$

$$\text{(4.1)} \qquad \begin{cases} \sum_n \left| \int_0^u h(n, u) du \right| < \infty, \\ \sum_m \left| \int_0^v g_r(m, v) dv \right| < \infty, \ (r = 1, 2, 3) \end{cases}$$

or, what is the same thing,

$$\sum_1 \equiv \sum_n \left| \frac{1}{\rho_n \rho_{n-1}} \sum_{k=0}^{n-1} \left(\frac{\rho_n}{k+1} - \frac{\rho_k}{n+1} \right) \frac{\sin (n-k)u}{(n-k) \log (n-k+1)} \right| < \infty,$$

$$\sum_2 \equiv \sum_m \left| \frac{\sin mv}{m \rho_m \log (m+1)} \right| < \infty,$$

$$\sum_3 \equiv \sum_m \left| \frac{1}{(m+1)\rho_m \rho_{m-1}} \sum_{l=1}^{m-1} \frac{\sin (m-l)v}{(l+1)(m-l) \log (m-l+1)} \right| < \infty,$$

$$\sum_4 \equiv \sum_m \left| \frac{1}{\rho_{m-1}} \sum_{l=1}^{m-1} \frac{\sin (m-l)v}{l(l+1)(m-l) \log (m-l+1)} \right| < \infty.$$

Denoting $\tau = [1/v,]$ we have

$$\text{(4.2)} \qquad \sum_2 = O\left[\sum_1^\infty \frac{1}{m \log^2 (m+1)} \right] = O(1).$$

$$\sum_3 \leqslant \sum_1^\tau \frac{1}{(m+1)\rho_m \rho_{m-1}} \left| \sum_{l=1}^{m-1} \frac{\sin (m-l)v}{(l+1)(m-l) \log (m-l+1)} \right|$$

$$+\sum_{\tau+1}^{\infty}\frac{1}{(m+1)\rho_m\rho_{m-1}}\left|\sum_{l=1}^{[m/2]-1}\frac{\sin(m-l)v}{(l+1)(m-l)\log(m-l+1)}\right|$$

$$+\sum_{\tau+1}^{\infty}\frac{1}{(m+1)\rho_m\rho_{m-1}}\left|\sum_{[m/2]}^{m-1}\frac{\sin(m-l)v}{(l+1)(m-l)\log(m-l+1)}\right|$$

$$=\underset{31}{\Sigma}+\underset{32}{\Sigma}+\underset{33}{\Sigma}. \tag{4.3}$$

Now since $|\sin(m-l)v| \leqslant (m-l)v$, we have

$$\underset{31}{\Sigma} \leqslant Av\sum_{1}^{\tau}\frac{1}{(m+1)\rho_m\rho_{m-1}}\cdot\sum_{l=1}^{m-1}\frac{1}{l+1}$$

$$=Av\sum_{1}^{\tau}\frac{1}{(m+1)\rho_m\rho_{m-1}}\cdot\rho_{m-1}=O(1). \tag{4.4}$$

Let δ be a fixed number greater than π, then by Abel's transformation, we have

$$\underset{32}{\Sigma}=\sum_{\tau+1}^{\infty}\frac{1}{(m+1)\,\rho_m\rho_{m-1}}\left|\sum_{l=1}^{[m/2]-1}\frac{1}{(m-l)\log(m-l+1)}\cdot\frac{\sin(m-l)v}{l+1}\right|$$

$$=O\left[\sum_{\tau+1}^{\infty}\frac{1}{(m+1)\,\rho_m\rho_{m-1}}\cdot\left(\log\frac{\delta}{v}\right)\cdot\frac{1}{\left(\left[\frac{m}{2}\right]+1\right)\log\left(\left[\frac{m}{2}\right]+2\right)}\right]$$

$$+O\left[\sum_{\tau+1}^{\infty}\frac{1}{(m+1)\,\rho_m\rho_{m-1}}\cdot\left(\log\frac{\delta}{v}\right)\left|\sum_{l=1}^{[m/2]-2}\Delta\left(\frac{1}{(m-l)\log(m-l+1)}\right)\right|\right]$$

$$=O\left(\log\frac{\delta}{v}\right)\sum_{\tau+1}^{\infty}\frac{1}{m^2\log^2 m}=O(1). \tag{4.5}$$

Also,

$$\underset{33}{\Sigma}=\sum_{\tau+1}^{\infty}\frac{1}{(m+1)\rho_m\rho_{m-1}}\cdot\left|\sum_{[m/2]}^{m-1}\frac{1}{l+1}\cdot\frac{\sin(m-l)v}{(m-l)\log(m-l+1)}\right|$$

$$=O\left[\sum_{\tau+1}^{\infty}\frac{1}{m^2\rho_m\rho_{m-1}}\cdot\sum_{[m/2]}^{m-1}\frac{1}{(m-l)\log(m-l+1)}\right]$$

$$=O\left[\sum_{\tau+1}^{\infty}\frac{\log\log m}{m^2\log^2 m}\right]=O(1). \tag{4.6}$$

Combining (4.3), (4.4), (4.5) and (4.6), we obtain

$$\underset{3}{\Sigma}=O(1). \tag{4.7}$$

Again

$$\underset{4}{\Sigma} \leqslant \sum_{1}^{\tau}\frac{1}{\rho_{m-1}}\left|\sum_{l=1}^{m-1}\frac{\sin(m-l)v}{l(l+1)(m-l)\log(m-l+1)}\right|$$

$$+\sum_{\tau+1}^{\infty}\frac{1}{\rho_{m-1}}\left|\sum_{l=1}^{[m/2]-1}\frac{\sin(m-l)v}{l(l+1)(m-l)\log(m-l+1)}\right|$$

$$+\sum_{\tau+1}^{\infty}\frac{1}{\rho_{m-1}}\left|\sum_{[m/2]}^{m-1}\frac{\sin(m-l)v}{l(l+1)(m-l)\log(m-l+1)}\right|$$

$$=\underset{41}{\Sigma}+\underset{42}{\Sigma}+\underset{43}{\Sigma}. \tag{4.8}$$

$$\underset{41}{\Sigma} = \sum_{1}^{\tau} \frac{1}{\rho_{m-1}} \left| \sum_{l=1}^{m-1} \frac{\sin (m-l)v}{l(l+1)(m-l) \log (m-l+1)} \right|$$

$$\leqslant Av \sum_{1}^{\tau} \frac{1}{\rho_{m-1}} \sum_{l=1}^{m-1} \frac{1}{l(l+1)} = O(1). \tag{4.9}$$

Since

$$\frac{1}{l(m-l)} = \frac{1}{ml} + \frac{1}{m(m-l)}, \text{ we have as in } \underset{32}{\Sigma}$$

$$\underset{42}{\Sigma} = \sum_{\tau+1}^{\infty} \frac{1}{\rho_{m-1}} \left| \sum_{l=1}^{[m/2]-1} \frac{\sin (m-l)v}{l(l+1)(m-l) \log (m-l+1)} \right|$$

$$\leqslant \sum_{\tau+1}^{\infty} \frac{1}{m\rho_{m-1}} \left| \sum_{l=1}^{[m/2]-1} \frac{\sin (m-l)v}{(l+1)(m-l) \log (m-l+1)} \right|$$

$$+ \sum_{\tau+1}^{\infty} \frac{1}{m\rho_{m-1}} \left| \sum_{l=1}^{[m/2]-1} \frac{\sin (m-l)v}{l(l+1) \log (m-l+1)} \right|$$

$$= O(1) + \sum_{\tau+1}^{\infty} \frac{1}{m\rho_{m-1}} \left| \sum_{l=1}^{[m/2]-1} \frac{1}{l \log (m-l+1)} \cdot \frac{\sin (m-l)v}{l+1} \right|$$

$$= O(1) + O\left[\sum_{\tau+1}^{\infty} \frac{1}{m\rho_{m-1}} \left(\log \frac{\delta}{v} \right) \cdot \frac{1}{\left(\left[\frac{m}{2} \right] - 1 \right) \log \left(\left[\frac{m}{2} \right] + 2 \right)} \right]$$

$$+ O\left[\sum_{\tau+1}^{\infty} \frac{1}{m\rho_{m-1}} \cdot \log \left(\frac{\delta}{v} \right) \cdot \sum_{l=1}^{[m/2]-2} \left| \Delta \left(\frac{1}{l \log (m-l+1)} \right) \right| \right]$$

$$= O(1) + O\left(\log \frac{\delta}{v} \right) \sum_{\tau+1}^{\infty} \frac{1}{m \log^2 m}$$

$$= O(1). \tag{4.10}$$

$$\underset{43}{\Sigma} = \sum_{\tau+1}^{\infty} \frac{1}{\rho_{m-1}} \left| \sum_{l=\left[\frac{m}{2}\right]}^{m-1} \frac{\sin (m-l)v}{l(l+1)(m-l) \log (m-l+1)} \right|$$

$$= O\left[\sum_{\tau+1}^{\infty} \frac{1}{\rho_{m-1}} \left| \sum_{\left[\frac{m}{2}\right]}^{m-1} \frac{1}{l(l+1)} \cdot \frac{1}{(m-l) \log (m-l+1)} \right| \right]$$

$$= O\left[\sum_{\tau+1}^{\infty} \frac{1}{m^2\rho_{m-1}} \cdot \sum_{\left[\frac{m}{2}\right]}^{m-1} \frac{1}{(m-l) \log (m-l+1)} \right]$$

$$= O\left[\sum_{\tau+1}^{\infty} \frac{\log \log m}{m^2 \log m} \right]$$

$$= O(1). \tag{4.11}$$

Combining (4.8), (4.9), (4.10) and (4.11), we obtain

$$\underset{4}{\Sigma} = O(1). \tag{4.12}$$

In view of (4.1), (4.2), (4.7), (4.12) and lemma 4, we have

$$\sum_m \sum_n \left| \sigma_{m,n} - \sigma_{m,n-1} - \sigma_{m-1,n} + \sigma_{m-1,n-1} \right| < \infty.$$

Similarly it can be shown that

$$\sum_n \left| \sigma_{m,n} - \sigma_{m,n-1} \right| < \infty,$$

$$\sum_m \left| \sigma_{m,n} - \sigma_{m-1,n} \right| < \infty.$$

This completes the proof of the theorem.

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References

Gergen, J. J. (1933). *Trans. Amer. math. Soc.*, **35**, 29–63.
Hardy, G. H., and Rogosinski, W. W. (1947). *Proc. Camb. phil. Soc.*, **43**, 10–25.
Herriot, J. G. (1942). *Trans. Amer. math. Soc.*, **52**, 72–94.
Sharma, P. L. (1958). *Proc. Amer. math. Soc.*, **9**, 979–986.
Timan, M. F. (1956). On absolute summability of Fourier series in two variables, *Soobshch. Akad. Nauk Gruz. SSR.*, **17**, 481–488.
Titchmarsh, E. C. (1939). The Theory of Functions. Oxford Univ. Press.
Varshney, O. P. (1959). *Proc. Amer. math. Soc.*, **10**, 784–789.
Young, W. H. (1913). *Proc. Lond. math. Soc.*, **11**, 133–184.